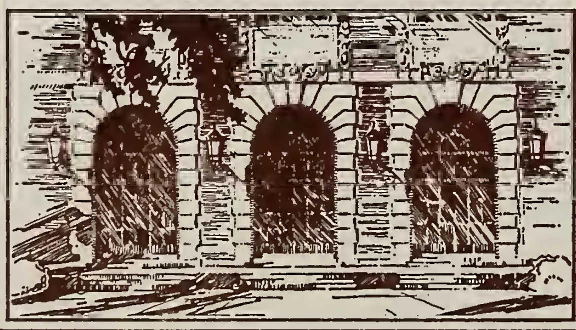

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ON THE UNIFORM APPROXIMATION OF A CLASS OF SINGULAR INTEGRAL EQUATIONS IN A HÖLDER SPACE

by

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December, 1972

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ON THE UNIFORM APPROXIMATION OF A CLASS OF SINGULAR INTEGRAL EQUATIONS IN A HÖLDER SPACE

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The application of the theory of collectively compact operator approximations to Fredholm integral equations gives theorems on the uniform convergence of, and error bounds for, the approximate solutions of these equations obtained by a direct method of numerical integration. It is the purpose of this paper to investigate the possibility of extending this concept to linear singular integral equations in a Hölder space.

In this paper it is shown that a sequence of approximate operators, obtained from the singular operator by a sequence of interpolatory quadrature rules, converges pointwise only on a certain subset of the Hölder space. Hence the convergence of the approximate solutions of the singular integral equation obtained by a direct method cannot be guaranteed by the collectively compact operator theory. However, it is shown that a direct approximation of the reduced equation does satisfy the hypotheses of the abstract operator approximation theory and that the sequence of approximate solutions converges in the desired space. It is concluded that this is an appropriate method of obtaining approximate numerical solutions to a linear singular integral equation.

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1. INTRODUCTION AND SUMMARY

1.1 Statement of the Problem

This paper is concerned with a method of obtaining approximate solutions of the linear singular integral equation discussed by Muskhelishvili [33]. This equation has the form

$$a(s)\varphi(s) + \int_L \frac{k(s,t)}{t-s}\varphi(t)dt = f(s)$$

where L is a smooth line and a(s), f(s), k(s,t) are given functions satisfying a Hölder condition.

There are a number of references to the approximate solution of singular integral equations (for example [18, p. 167], [23,37,41]) which generally involve methods (such as the method of moments) carried over from regular equations to singular equations. However these methods are not always the most convenient for numerical computations. This paper investigates the problem of extending, from Fredholm integral equations to singular integral equations, the method of approximating the integral equation directly by numerical integration.

Consider a Fredholm equation of the second kind

$$x(s) - \int_0^1 k(s,t)x(t)dt = y(s) \qquad , \; 0 \le s \le 1$$

where y(s), k(s,t) are continuous real or complex functions for $0 \le s, t \le 1$ and a continuous solution, x(s), is sought.

In a classical method of approximate solution based on numerical integration, this equation is replaced by an algebraic linear system

$$x_n(t_{ni}) - \sum_{j=1}^{n} w_{nj} k(t_{ni}, t_{nj}) x_n(t_{nj}) = y(t_{ni}) \qquad i=1,\ldots,n.$$

If this system has a solution, then an approximate solution $x_n(s)$, $0 \le s \le 1$, of the original integral equation can be obtained by interpolation.

The technique of replacing an integral equation by a finite system goes back at least to Fredholm, one of the founders of the theory. However there arises the question of the relationship of this approximate solution, $x_n(s)$, to the exact solution, $x(s)$, and the nature of the convergence of $x_n(s)$ to $x(s)$. Hilbert gave convergence proofs for approximate solutions using the rectangular quadrature formula. Other convergence theorems and error bounds for approximate solutions in particular cases and under various assumptions have been obtained by a number of investigators including Brakhage [10] and Kantorovich and Krylov [26]. Specific references to other investigations can be found in Anselone [5].

In a series of papers [1-8] beginning in 1964, P. M. Anselone and a number of co-workers were able to abstract the essence of these results, and to develop a general operator approximation theory for linear and non-linear operators on an arbitrary Banach space. The operator equation is regarded as being approximated by the same equation with an approximate operator replacing the exact operator. It is then shown that, under suitable conditions, the convergence of the sequence of operator approximations implies the uniform convergence of the solutions of these approximating equations to the solution of the original operator equation.

This collectively compact operator approximation theory is then applied to a wide variety of integral equations [5]. Anselone [2,4] showed that the abstract theory could be applied to an integral equation with a "mildly discontinuous" kernel by considering the analysis in the space, R,

of bounded, Riemann integrable functions with the supremum norm. Atkinson [7] further extended the class of integral operators to which the abstract theory is applicable by treating the case of an integral equation with a weakly singular kernel (that is with an integrable singularity). Anselone (in a lecture and also in a private communication) suggested that the collectively compact operator theory could also be superimposed on approximation techniques for singular integral operators. Such an extension to the case when the kernel is strongly singular involves the computation of singular integrals which are interpreted in the sense of the Cauchy principal value. Gakhov observes [18, p. 156] that methods of approximate computation of singular integrals have not been worked out in detail so far, either from the theoretical or from the practical point of view. Gakhov further notes [18, p. 167] that the methods of direct solution (in general approximate) of the complete singular integral equation have so far been little elaborated.

The theory of singular integral equations has practical applications in a number of important areas. Among these are a variety of problems in the theory of elasticity (Muskhelishvili [33, Chapter 13]), the problem of flow past an arc of given shape (Mikhlin [31, p. 325], Noble [36]), the theory of tides (Poincaré [39]) and more recently in the fields of dispersion relations and scattering theory.

1.2 Possible Algorithms for Direct Solution

Any such algorithm for direct solution will clearly be based on an approximate quadrature formula for principal value integrals. References to computational experience with such integrals are not widespread.

Voight [46] discusses the numerical solution of a Fredholm integral equation in which the known functions are themselves specified by principal

value integrals. These singular integrals are evaluated by using a Fourier series technique given by Collatz [11]. An appropriate transformation followed by the expansion of the numerator in a Fourier cosine series transforms the integral into a sum of principal value integrals whose values are known explicitly. Voight gives no discussion of the conditions under which this process is justified. He notes that, in the example given from annular airfoil theory, the computational results for the evaluation of the kernel of the free term were rather unsatisfactory. Noble [36, section 14] outlines essentially the same method for the numerical evaluation of this singular integral.

Delves [14] proposed a Newton-Cotes type method which is basically an application of the approximate product integration technique of Young [47], which was used by Atkinson [7] in the case of weakly singular integrands. The error bounds given by Delves depend on the numerator having a sufficient number of derivatives and on certain constants depending on the fixed value of the singularity. However it is not clear that such bounds will hold uniformly as the singularity varies in the range of integration.

Another general method of attacking this problem is the method of subtracting out the singularity. In this method the singular term is subtracted out of the integrand and this integral is evaluated analytically while the remaining integral (usually with an integrable singularity) is evaluated numerically. An example is the modified Gaussian rule mentioned by Delves [14]. A variation of this idea is the "method of symmetric pairing" suggested by Bareiss and Neuman [9]. In this method the function values of the integrand are "paired" for arguments that are symmetric with respect to the singularity. This again produces an integrand with an integrable singularity.

Finally we mention Stewart's method [42] which can be regarded as a "mixed" method. The singularity is subtracted out to produce a weakly singular integrand but then the numerator is interpolated, which is essentially an application of approximate product integration.

Hence the available methods can be divided into two basic classes:

(a) Approximate product integration techniques. These will include approximation of the numerator by polynomials (Delves), Fourier series (Collatz) or some other appropriate functions such as Chebyshev polynomials (compare Elliot's approach [16] for Fredholm equations).

(b) Reduction of the integrand to a function with an integrable singularity and the application of a direct quadrature formula.

In case (a) it is necessary to consider the convergence of the sequence of interpolating polynomials (see section 2.2) and also the convergence of a sequence of principal value integrals (section 3.4.2). In case (b) the convergence of quadrature formulas for unbounded integrands must be considered (section 2.5).

In the application of these quadrature methods to the direct solution of singular integral equations, the first class seems to be the most natural. An example is Gabdulhaev's investigation [17] of the method of constructing solutions by polygonal approximation. Gabdulhaev's paper does not consider the convergence of this quadrature process and does not give a numerical example to illustrate the method. However, there is a theoretical justification of the convergence of the approximate solutions of the singular integral equation in a Hölder space, obtained by this method. This justification is based generally on Kantorovich's general theory of approximate methods [25, chapter 14]. The paper and its implicit assumptions are discussed in section 3.5.

Bareiss and Neuman apply their method of the second class to the solution of the homogeneous singular integral equation arising from Milne's problem in radiative transfer, but the authors do not attempt a theoretical justification of their results.

1.3 Convergence Theory

It was noted above that the integral equation is replaced by an algebraical linear system by means of a numerical integration formula. The solution of this linear system, or approximate equation, can be regarded as an approximate solution to the original equation. However it is necessary to ensure that such an approximate solution does converge to the exact solution in some appropriate sense. In order to avoid the necessity of considering individual cases, unified theories of approximation methods have been constructed in a functional analysis framework. We are primarily concerned with Anselone's collectively compact operator approximation theory but also consider Kantorovich's general theory of approximation methods.

1.3.1 Anselone Theory [5]

Let X denote an arbitrary real or complex Banach space. The closed unit ball in X will be denoted by $\mathcal{B} = \{x \in X : \|x\| \leq 1\}$. Denote by $[X]$ the Banach space of bounded linear operators $T: X \to X$, with the usual operator norm, $\|T\| = \sup_{x \in \mathcal{B}} \|Tx\|$. Convergence in norm of the sequence of operators is written $\|T_n - T\| \to 0$ and the notation $T_n \to T$ will be used for pointwise convergence, i.e., $\|T_n x - Tx\| \to 0$ as $n \to \infty$ for each $x \in X$.

In the standard approximation theory (for example, Taylor [43], or Kantorovich [25]) the operator T is uniformly approximated in norm by a sequence of approximating operators T_n. Then based on the well-known Neumann series representation for $(I-T)^{-1}$ the following basic proposition is obtained.

Proposition 1.3.1 [5, prop. 1.4]

Let $T, T_n \in [X]$ and $\|T_n - T\| \to 0$. Then there exists $T^{-1} \in [X]$ iff for some N and all $n \geq N$ there exist $T_n^{-1} \in [X]$ bounded uniformly in n, and in this case $\|T_n^{-1} - T^{-1}\| \to 0$ as $n \to \infty$.

However, the condition of norm convergence is not met in many cases of practical interest, and in particular in the case of the approximation of integral operators by quadrature formulas. But Anselone shows how the weaker hypothesis of pointwise convergence, $T_n \to T$, combined with certain compactness conditions can be used to deduce an analogous theorem giving pointwise convergence of the inverse operators together with useful error bounds. The following concepts are needed where X has been assumed complete.

Definition: [5]

An operator $K \in [X]$ is compact (or completely continuous) iff the set $K\mathcal{B}$ is precompact.

Definition: [5]

A set of operators $\mathcal{K} \subset [X]$ is collectively compact iff the set $\mathcal{K}\mathcal{B} = \{Kx : K \in \mathcal{K}, x \in \mathcal{B}\}$ is precompact.

With these definitions, the development of the theory may be summarized by stating the following convergence theorem whose proof may be found in the papers of Anselone, particularly [3,5].

Theorem 1.3.2 [5, Theorem 1.11]

Let K, $K_n \in [X]$ and suppose that the approximate operators K_n satisfy

1. Pointwise convergence, $K_n \to K$
2. $\{K_n : n \geq 1\}$ is collectively compact, which together imply that
3. K is compact

Suppose also that $(I-K)^{-1}$ exists and define

$$\Delta_n = \|(I-K)^{-1}\| \cdot \|(K_n-K)K_n\|$$

Then $\Delta_n \to 0$ and thus for all n sufficiently large $\Delta_n < 1$, in which case $(I-K_n)^{-1}$ exists and is bounded uniformly in n,

$$\|(I-K_n)^{-1}\| \le \frac{1+\|(I-K)^{-1}\| \cdot \|K_n\|}{1-\Delta_n},$$

and

$$\|x_n - x\| = \|(I-K_n)^{-1}y - (I-K)^{-1}y\| \le \frac{\|(I-K)^{-1}\| \cdot \|K_n y - Ky\| + \Delta_n \|(I-K)^{-1}y\|}{1-\Delta_n} \to 0$$

1.3.2 Kantorovich Theory [25, Chapter 14]

Kantorovich's general theory for equations of the second kind projects the exact equation $Kx \equiv x - Hx = y$ in the normed space X (not necessarily assumed to be complete) into an approximate equation $\tilde{K}\tilde{x} \equiv \tilde{x} - \tilde{H}\tilde{x} = \tilde{y}$ in the complete subspace, $\tilde{X}$, of X. The theory involves the following three conditions connecting the normed space X and the space $\tilde{X}$ of the approximate equation, where $\tilde{X}$ is the projection of the space X by the linear operator P, and $\tilde{H}$ is the linear operator in $\tilde{X}$ corresponding to the general operator H in X.

I: (Condition that H and $\tilde{H}$ be neighboring operators)
For every $\tilde{x} \in \tilde{X}$, $\|PH\tilde{x} - \tilde{H}\tilde{x}\| \le \eta \|\tilde{x}\|$

II: (Condition for the elements of the form Hx to be approximated closely by elements of $\tilde{X}$). For every $x \in X$, there exists $\tilde{x} \in \tilde{X}$ such that $\|Hx - \tilde{x}\| \le \eta_1 \|x\|$

III: (Condition for the close approximation of the free term of the equation) There exists $\tilde{y} \in \tilde{X}$ such that $\|y - \tilde{y}\| \le \eta_2 \|y\|$

The following theorem gives conditions for the convergence of the sequence $\{\tilde{x}_n^*\}$ of approximate solutions to the exact solution x^*.

Theorem 1.3.3 [25, Theorem 3, p. 549]

If the following conditions are satisfied:

1. The operator K has a linear inverse,
2. The existence of a solution of $\tilde{K}\tilde{x}=\tilde{y}$ for every $\tilde{y}\epsilon\tilde{X}$ implies its uniqueness,
3. The conditions I, II, III are satisfied for all n and $\eta\to 0$, $\eta_1\|P_n\|\to 0$, $\eta_2\|P_n\|\to 0$ as $n\to\infty$,

then the approximate equations are soluble for sufficiently large n and the sequence of approximate solutions converges to the exact solution:

$$\|x^*-\tilde{x}_n^*\| \le Q_o\eta + Q_1\eta_1\|P_n\| + Q_2\eta_2\|P_n\| \to 0 \text{ as } n\to\infty,$$

where Q_0, Q_1, Q_2 are constants.

Thus Kantorovich considers the solution of an approximate (or projected) equation in a convenient "approximation subspace" of the space of the exact solution.

1.4 Summary of the Paper

It has been noted that the pointwise convergence of the approximate operator basically depends on the convergence properties of the approximate quadrature scheme that is applied. Thus in order to weaken the sufficient conditions (i.e., the hypotheses of theorem 1.3.2) for the applicability of the theory to the solution of integral equations, we consider the convergence of methods of numerical integration and the convergence of the resulting approximate integral operators.

In Chapter 2 the convergence of approximate integration formulas is considered, using the convergence properties of the interpolation polynomials and the modulus of continuity of the integrand. These results are applied to the integral operator with a continuous kernel. It is shown that, for an interpolatory quadrature rule, the sufficient conditions for the application of the uniform approximation theory in the space C may be recast as conditions on the partial moduli of continuity of the kernel of the integral operator. To extend the theory to the case of a mildly discontinuous kernel Anselone [2] carried out the analysis in the space R of proper Riemann integrable functions with the supremum norm. In extending the above approach to the space R we obtain the more general sufficient conditions that the sets of functions obtained by regarding the kernel as a function of one variable only, are both regular sets. In the case of a weakly singular kernel, the difficulties with the numerical quadrature of singular integrands (Davis and Rabinowitz [13]) preclude the application of a direct quadrature rule. However the application of a rule of approximate product integration (Atkinson [7]), regarding the singular part of the integrand as part of the operation of integration, reduces this case to one of the above cases.

This approach is applied to the case of a singular integral operator in Chapter 3. It is shown that an interpolating polygon of a function in a Hölder space converges to its generating function only on a certain subset of the Hölder space. Although the singular integral operator is a bounded operator on the Hölder space, it is demonstrated that a sequence of approximate operators obtained by approximate product integration converges pointwise only on an equi-Hölder continuous subset of the Hölder space. Hence it is concluded that the uniform approximation theory cannot be applied in this case since the hypothesis of pointwise convergence is not satisfied.

Gabdulhaev's direct method [17] for solving singular integral equations in a Hölder space is considered in section 3.5. It is shown that his discussion of the convergence of the method cannot be considered as satisfactory since the convergence obtained in the larger Hölder space does not imply convergence in the solution space. Gabdulhaev's approach is based on Kantorovich's general theory of approximate methods. This theory is considered in section 3.6 for the case of an integral equation. It is shown that the conditions for the application of the convergence theorem may be reduced to the same as those obtained for Anselone's theory in Chapter 2. From this analysis it is noted that the implicit assumptions made by Gabdulhaev are not necessarily justified.

In Chapter 4 it is shown that a uniform approximation may be obtained by considering the reduced equation. It is noted that the hypotheses on the kernel need to be modified slightly to obtain a consistent equation. The reduced equation equivalent to the singular equation may be considered in the space C. In section 4.3 we define approximate operators by approximate product integration of the component singular operators. These approximations involve function evaluations only of the given functions. It is shown that these approximations are collectively compact and converge pointwise in C. Hence the Anselone theory is applicable and it is shown that the approximate solutions converge uniformly to the correct solution although error bounds are not easily computed.

2. THE UNIFORM APPROXIMATION THEORY FOR INTEGRAL EQUATIONS

2.1 Introduction

In order to determine whether Anselone's uniform approximation theory may be extended to the case of the singular integral equation, the basis for the sufficient conditions in Theorem 1.3.2 is examined. It is shown that these conditions may be recast so as to depend on the properties of the kernel of the integral operator. This suggests the argument necessary to deal with the operator with a singular kernel.

The abstract theory requires certain compactness conditions on the operators. This is to compensate for the inadequacy of the weaker condition of pointwise convergence of the approximate operators obtained by quadrature. Anselone's theory obtains uniform convergence of the operator sequences by restricting the class of admissible functions in the space, C, of continuous functions to the bounded, equicontinuous set $K\mathcal{B}$. Since the operator K is bounded by hypothesis, Anselone's theory in the space C effectively requires only that the operator sequence converges uniformly on an equicontinuous set of functions, and that the set $K\mathcal{B}$ be equicontinuous. In order to generalize this result it is necessary to consider what are the equivalent conditions on other spaces. Thus the convergence properties of a sequence of approximate quadrature formulas for less smooth integrands are investigated.

The uniform or supremum norm is also the norm for the space, R, of bounded Riemann integrable functions. Suppose that the Fredholm integral equation in C, with a continuous kernel and a continuous given function, is regarded as an equation in the space R. Then we may seek a solution in the space R and such solution is automatically continuous. This follows since for an equation $x(s)-\int k(s,t)x(t)dt = f(s)$, if $f\in C$ and the kernel k is such

that $\int |k(s,t)| dt = g(s) \epsilon C$, then $x \epsilon C$. In particular equations whose kernels are mildly discontinuous [2] or weakly singular [7] satisfy this condition.

If $g(s) \epsilon R$ or $f(s) \epsilon R$ then the equation is in the space R and the solution, if it exists, is also in R. Thus by obtaining the conditions for the convergence of the approximate quadrature formulas in the space R, the application of the general theory is extended to equations in the space R (section 2.4). It is clear however, that we cannot infer by this argument a bounded integrable solution if k is a singular kernel since the Cauchy principal value is not absolutely integrable.

The results of this investigation provide a basis for the consideration of the convergence of approximations to singular operators in a Hölder space in the following chapter.

2.2 Convergence of Formulas for Approximate Integration

It was observed by Krylov [27] that the convergence of an interpolatory quadrature rule is closely related to the convergence of the interpolation process itself. In particular if the interpolation process converges uniformly in the range of integration then the quadrature process will also converge. Krylov obtains regions of holomorphy of the function in order to obtain uniform convergence of the interpolations. However we are more concerned here with obtaining convergence results to enable the identification of those sets of functions over which the sequence of quadrature functionals will converge uniformly. Anselone's theory is basically concerned with this property since the convergence of quadrature functionals is not uniform on bounded sets in C.

Consider a sequence of Lagrange interpolating polynomials on an arbitrary sequence of nodes in a closed interval. The basic convergence

result for a continuous function $x(t)$ is given by

Lemma 2.2.1 (Natanson [35, p. 47])

If $x(t) \in C[a,b]$ and E_n is its best approximation by polynomials of degree not greater than n, then the interpolation polynomial $L_n x$ satisfies

$$\|L_n x - x\| \leq \|L_n\| \cdot E_{n-1}$$

where $\|L_n\|$ is the norm of the Lagrange interpolation operator.

The class of functions for which the sequence converges uniformly includes entire functions [35, p. 10]. But it is substantially more restricted than the class of continuous functions since the Faber-Bernstein theorem [35, p. 27] gives the inequality $\|L_n\| > (\log n)/8\sqrt{\pi}$ in general.

However by making the restriction to piecewise polynomial interpolation a more useful result may be obtained.

Lemma 2.2.2

If L_m denotes the Lagrange interpolation operator of degree m, and P_n^m denotes the piecewise m-th degree polynomial operator over n subintervals, then

$$\|P_n^m x - x\| \leq m\|L_m\|\omega(x,\delta)$$

where $\omega(x,\delta)$ is the modulus of continuity of $x \in C[a,b]$ and $\delta = 0(\frac{1}{n})$ is the norm of the net of interpolation points.

Proof:

For an arbitrary value t in one of the n subintervals

$$|P_n^m x(t) - x(t)| = |L_n x(t) - x(t)| = |\sum_{k=0}^{m} x(t_k)\ell_k(t) - x(t)|$$

where
$$\ell_k(t) = \frac{(t-t_0)\ldots(t-t_{k-1})(t-t_{k+1})\ldots(t-t_m)}{(t_k-t_0)\ldots(t_k-t_{k-1})(t_k-t_{k+1})\ldots(t_k-t_m)}$$

Thus since $\sum_{k=0}^{m} \ell_k(t) = 1$, the right side becomes

$$= \left| \sum_{i=0}^{m} (x(t_k)-x(t))\ell_k(t) \right|$$

$$\leq \omega(x,m\delta) \sum_{k=0}^{m} |\ell_k(t)| \qquad \text{where } \delta = 0(\tfrac{1}{n})$$

$$\leq m \, \|L_m\| \, \omega(x,\delta).$$

Although the Faber theorem implies that a uniformly convergent interpolation process for every continuous function cannot be obtained, the situation is more favorable with respect to convergence in the mean:

Lemma 2.2.3 (Natanson [35, p. 56])

For an arbitrary orthogonal polynomial system with weight function p(t), the inequality

$$\int_a^b p(t)\{L_n x(t)-x(t)\}^2 dt \leq C^2\omega^2(x,\tfrac{1}{n-1})\int_a^b p(t)dt$$

holds for every continuous function x(t), where C is an absolute constant.

These results may then be applied to obtain convergence results for the associated interpolatory quadrature rules. Approximating the integrand by a sequence of interpolations of increasing order leads to a sequence of Newton-Cotes formulas.

Denote $\Phi x = \int_0^1 x(t)dt$ and the approximate formula $\Phi_n x = \Phi L_n x = \int_0^1 L_n x(t)dt$. Thus

$$|\Phi x-\Phi_n x| = |\Phi(x-L_n x)| \leq \|\Phi\| \, \|x-L_n x\|.$$

It is well known (Krylov [27]) that this sequence does not converge for $x \in C$ since the coefficients are asymptotically unbounded. However from Lemma 2.2.1 we may deduce uniform convergence over a class of sufficiently smooth functions. For example (Nathanson [35]) if the nodes are the zeros of the Chebyshev polynomial and the function x belongs to the Dini-Lipschitz class $\{x : \omega(x,\delta)\log\delta \to 0 \text{ as } \delta \to 0\}$, then the interpolation sequence converges uniformly. Thus in this case, using the Jackson theorem, we have

$$|\Phi x - \Phi_n x| \leq \|\Phi\| \|x - L_n x\| \leq C.\|\Phi\|.\omega(x,\tfrac{1}{n})\log\tfrac{1}{n}$$

and the quadrature converges uniformly over the Dini-Lipschitz class.

The interpolation quadrature rule associated with a polygonal approximation is simply the compound trapezian rule. Similarly, piecewise quadratic interpolations will lead to a compound Simpson's rule, and more generally, piecewise interpolations by polynomials of the m-th degree leads to a corresponding compound Newton-Cotes rule.

<u>Theorem 2.2.4</u>

A sequence of composite, interpolatory quadrature functionals converges uniformly on each equicontinuous set.

<u>Proof</u>: Consider a composite quadrature rule based on piecewise interpolation by polynomials of the m-th degree. Then

$$\begin{aligned} |\Phi x - \Phi_n x| &= |\Phi(x - P_n^m x)| \leq \|\Phi\|.\|x - P_n^m x\| \\ &\leq \|\Phi\|.m.\|L_m\|.\omega(x,\delta) \qquad \text{by lemma 2.2.2} \end{aligned}$$

Since a subset, $E \subset C[0,1]$ is equicontinuous iff $\sup_{x \in E} \omega(x,\delta) \to 0$ as $\delta \to 0$, the result follows since $\delta = 0(\frac{1}{n})$.

The theorem applies equally to a composite rule involving a weight function p(t). Let $Qx = \int_0^1 p(t)x(t)dt$. Interpolating the operand x(t) as before we obtain

$$Q_n x = QP_n^m x = \int_0^1 p(t)P_n^m x(t)dt$$

which is the form of approximate product integration used by Young [47] and Atkinson [7], where $p(t)$ is supposed to be Lebewque integrable.

Corollary 2.2.5

A sequence of approximate product integration functionals converges uniformly over an equicontinuous set.

Proof:

$$|Qx-Q_n x| = |Q(x-P_n{}^m x)| \leq \|Q\|.\|x-P_n{}^m x\| \text{ where } \|Q\| = \int_0^1 |p(t)|dt.$$

The result follows by an application of lemma 2.2.2 as in the theorem.

A particular case arises when the nodes of the Lagrange interpolation operator, L_n, are chosen as the roots of an orthogonal polynomial system with respect to the weight $p(t)$. This gives the Gaussian quadrature formulas for which pointwise convergence for each $x\epsilon C$ is usually shown by utilizing the Weierstrass approximation theorem (for example Natanson [35], Davis and Rabinowitz [12]).

Theorem 2.2.6

Let $\{Q_n\}$ denote a sequence of Gaussian quadrature rules. Then Q_n converges uniformly to the functional Q on each equicontinuous set.

Proof:

$$\begin{aligned} |Qx-Q_n x| &= \left|\int_a^b p(t)\{x(t)-L_n x(t)\}dt\right| \\ &\leq \left\{\int_a^b |p(t)|dt\right\}^{1/2}\left\{\int_a^b |p(t)|\;|x(t)-L_n x(t)|^2 dt\right\}^{1/2} \\ &\qquad \text{by the Holder inequality} \\ &\leq C.\omega(x,\tfrac{1}{n-1}) \int_a^b |p(t)|dt \qquad \text{by lemma 2.2.3.} \end{aligned}$$

Thus $|Qx-Q_n x| \leq C.\|Q\| \sup_{x\epsilon E} \omega(x,\frac{1}{n-1})$, and the result follows as in the previous theorem.

Davis and Rabinowitz [12] also show that the convergence of the quadrature formulas is also valid if $x(t)$ is a bounded Riemann integrable function and if the formula has positive weights. (Gaussian rules and low

order composite Newton-Cotes formulas). That is $\Phi_n x \to \Phi x$ as $n \to \infty$ for each $x \in R$, where R is the space of properly Riemann integrable functions on a finite interval with the sup norm, and Φ_n is a positive linear functional. Hence to extend the above results to the space R, we define the concept of a regular set, which gives a kind of uniform integrability condition.

Definition:

A subset $S \subset R$ is regular iff for each $x \in S$ and each $m=1,2,\ldots$, there exist x_m and x^m in C such that $x_m \le x \le x^m$ and $\Phi(x^m - x_m) \to 0$ uniformly for $x \in S$ as $m \to \infty$, and, for each fixed m, the sets $S_m = \{x_m : x \in S\}$, $S^m = \{x^m : x \in S\}$ are equicontinuous.

Note that this definition is a weaker form of Anselone's definition of a regular set in that the sets S^m, S_m are required only to be equicontinuous and not necessarily precompact. However it still follows that an equicontinuous set is regular but not necessarily conversely. (For example, the set of all characteristic functions of intervals on $[0,1]$). A bounded set may not be regular. The properties of regular sets given by Anselone [5] are unchanged by this modified definition.

Theorem 2.2.7

A sequence of positive, interpolatory quadrature functionals converges uniformly on each regular set $S \subset R$.

Proof:

Since $x \in R$, for each $m=1,2,\ldots$, there exist x^m, $x_m \in C$ such that $x_m \le x \le x^m$ and $\Phi(x^m - x_m) \to 0$ as $m \to \infty$. Since Φ, Φ_n are positive functionals

$$\Phi(x_m - x^m) + \Phi_n x_m - \Phi x_m \le \Phi_n x - \Phi x \le \Phi_n x^m - \Phi x^m + \Phi(x^m - x_m)$$

If $x \in S$, a regular set, then $\Phi(x^m - x_m) \to 0$ uniformly for $x \in S$. Further an

application of theorem 2.2.4 or theorem 2.2.6, as appropriate, implies that $\Phi_n \to \Phi$ uniformly on the equicontinuous sets S^m, S_m. Hence the inequality above implies $\Phi_n \to \Phi$ uniformly for $x \in S$.

Although the above theorems are implicitly presented in terms of real functions, the results extend immediately to complex functions since many assertions reduce to the corresponding real cases (Anselone [5]).

In practical applications it will generally be more convenient to use a simple composite quadrature rule, although the theorems of this section indicate a more general application. Thus the operator P_n will refer to any of the polynomial interpolation operators applicable to these theorems. For definiteness it will often be considered as a piecewise polynomial operator of some fixed degree (often linear).

2.3 Integral Operators

These results are now applied to the case of an integral operator, where by an approximate quadrature rule it is to be understood a rule of the type discussed in the preceding section.

Consider initially the integral operator K, with continuous kernel $k(s,t)$, as an operator from C into C. Write

$$y = Kx \text{ where } y(s) = \int_0^1 k(s,t)x(t)dt.$$

We may write $k_s(t) = k(s,t) = k_t(s)$ where k_s, $k_t \in C$. We will use the convention that the subscript s will refer to the first variable of the kernel k and the subscript t will refer to the second. Thus $Kx(s) = \Phi(k_s x)$ and $K_n x(s) = \Phi_n(k_s x)$ where Φ_n is the approximate quadrature rule as before.

It can be easily verified that the function $k(s,t)$, continuous as a function of two variables on the closed unit square, implies that the sets $\{k_s : 0 \le s \le 1\}$, $\{k_t : 0 \le t \le 1\}$ are bounded and equicontinuous. The converse result is not true in general since a function of two variables may be continuous in each variable without being continuous in both.

Definition: (Timan [44])

The partial modulus of continuity, $\omega_s(\delta)$ of the function $k(s,t)$ regarded as a function of s is defined by

$$\omega_s(\delta) = \sup_{\substack{0<s,t<1 \\ |h|<\delta}} |k(s+h,t)-k(s,t)| = \sup_t \omega(k_t,\delta)$$

The function $\omega_s(\delta)$ is a modulus of continuity and the partial modulus of continuity, $\omega_t(\delta)$, is defined similarly. It follows that if the partial moduli of continuity exist as simple moduli of continuity then the sets $\{k_s\}$, $\{k_t\}$ are bounded and equicontinuous.

In the previous section it was noted that the sequence of approximate quadrature rules with integrand x converges pointwise if $\|x-P_n x\| \to 0$, and uniformly on each equicontinuous subset.

Lemma 2.3.1

The sequence $\{K_n\}$, of approximate operators obtained by approximate quadrature, converges pointwise to the integral operator K if the partial modulus of continuity of the kernel, $\omega_t(\delta)$, exists as a simple modulus of continuity.

Proof:

By lemma 2.2.2 we have

$$\begin{aligned}\|k_s x - P_n k_s x\| &\le C\omega(k_s x,\delta)\\ &\le C[\|x\|\omega(k_s,\delta) + \|k_s\|\omega(x,\delta)]\\ &\le C[\|x\| \sup_s \omega(k_s,\delta) + \sup_s\|k_s\|\omega(x,\delta)]\\ &= C\|x\|\omega_t(\delta) + C.M.\omega(x,\delta) \qquad (2.3.1)\end{aligned}$$

where C is a constant depending on the fixed degree of the interpolating polynomial and M is the absolute bound of the kernel k(s,t). Hence, arguing as in theorem 2.2.4, for the integral operator K,

$$\begin{aligned}|Kx(s)-K_n x(s)| &= |(\Phi-\Phi_n)(k_s x)| = |\Phi\{k_s x - P_n k_s x\}|\\ &\le \|\Phi\| . \|k_s x - P_n k_s x\|\\ &\le C.\|\Phi\| .[\|x\|\omega_t(\delta) + M\omega(x,\delta)]\end{aligned}$$

by the inequality (2.3.1).

Thus for each fixed $x \epsilon C$, the right side converges uniformly with respect to s if $\{k_s\}$ is an equicontinuous set (i.e., if $\omega_t(\delta)$ exists as a simple modulus of continuity). It is concluded that $\|Kx(s)-K_n x(s)\| \to 0$ as $n \to \infty$ for each $x \epsilon C$ if $\omega_t(\delta) \to 0$.

From the proof of this lemma it is also clear that this operator convergence will be uniform on each bounded, equicontinuous set, E, of operands. While the argument of theorem 2.2.4 again implies that the linear functionals Φ_n converge uniformly on the set $\{k_s x: 0 \le s \le 1,\ x \epsilon E\}$, whether or not this set is bounded, it does not follow in the above proof that the operators K_n converge uniformly on the equicontinuous set E, whether or not this is

bounded. However since bounded linear operators are being considered, a precompact range will be obtained in any case. It is concluded from the lemma that the equicontinuity of the set $\{k_s : 0 \le s \le 1\}$ is sufficient for the pointwise convergence of the operators K_n to K.

Lemma 2.3.2

The integral operator K is compact if the partial modulus of continuity of the kernel, $\omega_s(\delta)$, exists as a simple modulus of continuity.

Proof:

The operator K is compact if $\{Kx : x \in \mathcal{B}\}$ is precompact, where $\mathcal{B}$ is the unit ball. Thus for $|s_1 - s_2| < \delta$

$$\begin{aligned} |Kx(s_1) - Kx(s_2)| &\le \int_0^1 |k(s_1,t) - k(s_2,t)| \,.\, |x(t)|\,dt \\ &\le \|x\| \,.\int_0^1 \omega(k_t,\delta)\,dt \le \|x\|\,.\,\omega_s(\delta). \end{aligned}$$

Hence $\{Kx : x \in \mathcal{B}\}$ is equicontinuous and since the operator K is clearly bounded, the result follows.

Consider a collection of subsets of the space C.

Definition: The collection of sets of functions $\{E_n\}$ is collectively equicontinuous iff $\sup_n \sup_{x \in E_n} \omega(x,\delta) \to 0$ as $\delta \to 0$.

Definition: The collection of sets of functions $\{E_n\}$ is called collectively precompact if $\{E_n\}$ is uniformly bounded and collectively equicontinuous.

This follows from the Arzela-Ascoli lemma and it follows analogously, that the set of operators $\{K_n : n \ge 1\}$ is collectively compact iff the collection $\{K_n \mathcal{B} : n \ge 1\}$ is collectively precompact. This gives the same definition of a collectively compact set of operators in [C] as the more general definition given in Section 1.3.1.

Lemma 2.3.3

Let $\{K_n\}$ be the sequence of operators obtained from the integral operator K by a sequence of approximate quadrature rules. Then $\{K_n\}$ is collectively compact if the partial modulus of continuity of the kernel, $\omega_s(\delta)$, exists as a simple modulus of continuity.

Proof:

Since the sequence of approximate quadrature rules considered in section 2.2 is pointwise convergent, the Banach-Steinhaus Theorem [25, p. 252] implies that the sequence of operators is uniformly bounded. Thus $\|\Phi_n\| \leq B<\infty$ for some B and all n. Thus $\|K_n x\| = \sup_s|\Phi_n(k_s x)| \leq \|\Phi_n\|.\|x\|\sup_s\|k_s\| \leq B.M.\|x\|$. Hence $\{K_n\mathcal{B}: n\geq 1\}$ is uniformly bounded.

Note that a definite value for the bound B may be obtained directly if required. In the case of a composite rule (cf. corollary 2.2.5) it is easy to show that $B \leq \|Q\|.\|L_m\|$ where L_m is the fixed interpolation operator of degree m. In the case of a Gaussian rule, applying the Hölder inequality as in Theorem 2.2.6 and noting the orthogonality of the basis polynomials $l_k(t)$, shows that $B \leq \|Q\|$.

For two arbitrary points s_1, s_2 such that $|s_1-s_2|<\delta$,

$$|K_n x(s_1)-K_n x(s_2)| = |\Phi_n(k_{s_1}x-k_{s_2}x)| \leq \|\Phi_n\|.\|x\|\sup_s|k(s_1,t)-k(s_2,t)|$$

$$\leq B.\|x\|\sup_t \omega(k_t,\delta) \leq B.\|x\|\omega_s(\delta).$$

Hence $\{K_n x: \|x\|\leq 1, n\geq 1\}$ is collectively equicontinuous if $\omega_s(\delta)\to 0$ as $\delta\to 0$. Thus $\{K_n\mathcal{B}: n\geq 1\}$ is collectively precompact and $\{K_n: n\geq 1\}$ is a collectively compact set of operators.

From the previous two lemmas the same condition on the partial modulus of continuity of the kernel implies both compactness of the operator K and collective compactness of the set of approximate operators $\{K_n\}$. Thus in summary we obtain the theorem

Theorem 2.3.4

If K is an integral operator on C with a continuous kernel k(s,t), and $\{K_n\}$ is a sequence of approximating operators obtained by applying a sequence of approximate quadrature rules, then it is sufficient for the application of the collectively compact operator theory that the two partial moduli of continuity of the kernel exist as simply moduli of continuity.

Proof:

1. $\omega_s(\delta)\to 0$ as $\delta\to 0$ implies that K is compact (lemma 2.3.2) and $\{K_n : n\geq 1\}$ is collectively compact (lemma 2.3.3)

2. $\omega_t(\delta)\to 0$ as $\delta\to 0$ implies that $K_n\to K$ pointwise as $n\to\infty$ (lemma 2.3.1)

The result follows immediately by theorem 1.3.2. Since k is (uniformly) continuous on the unit square its modulus of continuity, $\omega(k;\delta_1,\delta_2)\to 0$ as $\delta_1,\delta_2\to 0$. Since $\max\{\omega_s(\delta_1),\omega_t(\delta_2)\} \leq \omega(k;\delta_1,\delta_2)$, the theorem follows for a uniformly continuous kernel.

It should be noted that this theorem for bounded, integral operators requires that both the sets $\{k_s : 0\leq s\leq 1\}$, $\{k_t : 0\leq t\leq 1\}$ be equicontinuous. It was observed that if k(s,t) is uniformly continuous then this condition is satisfied. It might be suspected that in principle it would be sufficient to require k to be continuous in each variable separately without being continuous in both. However from Timan [44] we have the relation

$$\max\{\omega_s(\delta_1),\omega_t(\delta_2)\} \leq \omega(k;\delta_1,\delta_2) \leq \omega_s(\delta_1) + \omega_t(\delta_2)$$

Thus if both sets are equicontinuous then $\omega_s(\delta_1)\to 0$ and $\omega_t(\delta_2)\to 0$ and hence $\omega(k;\delta_1,\delta_2)\to 0$ as $\delta_1,\delta_2\to 0$. That is, k is uniformly continuous as a function of two variables.

2.4 Equations in the Space R

The last theorem does not reveal a new result but it does shift the emphasis for sufficient conditions onto the properties of the kernel k. In this context we may investigate sufficient conditions under weaker conditions on the kernel. Thus an extension to the results obtained by Anselone [2] for the case of a mildly discontinuous kernel is suggested. The analysis, in this case, is carried out in the space R.

It is shown [2] that if the kernel k(s,t) is uniformly t-integrable then $KR \subset C$, K is compact (in the space C) and $\{K_n\}$ is collectively compact. Thus all the conditions for the uniform approximation theory to be applied are satisfied.

These conditions can be weakened to require only that K maps R into R. A parallel approach to that used in the last section for a continuous kernel suggests that conditions might be obtained which are similar for both variables s,t (compare theorem 2.3.4).

By theorem 2.2.7 a sequence of positive, interpolatory quadrature functionals converges uniformly on each regular subset.

Lemma 2.4.1

The sequence $\{K_n\}$ of approximate operators obtained by applying positive, approximate quadrature rules converges pointwise to the integral operator K if $\{k_s : 0\le s\le 1\}$ is a regular set.

Proof:

If $\{k_s\}$ is regular then $\{k_s x : 0\le s\le 1\}$ is a regular set for each fixed $x\in R$. Thus

$$\|K_n x-Kx\| = \sup_s |(\Phi_n-\Phi)(k_s x)| \to 0 \text{ as } n\to\infty$$

by theorem 2.2.7.

If the analogy with the continuous case were to be continued we would require $\{k_t\}$ regular to obtain a condition analogous to the compactness of the operator K. This symmetry is implicitly required to obtain the necessary convergence condition $\|(K_n-K)K\|\to 0$ since

$$\|K_nK-K^2\| = \sup_s \int_0^1 |k_{n2}(s,t)-k_2(s,t)|dt$$

$$\text{where } k_2(s,t) = \int_0^1 k(s,u)k(u,t)du = \Phi(k_s k_t)$$

$$\text{and } k_{n2}(s,t) = \sum_{i=1}^n w_{ni}k(s,u_{ni})k(u_{ni},t) = \Phi_n(k_s k_t)$$

Thus $\|(K_n-K)K\| = \sup_s \int_0^1 |(\Phi_n-\Phi)(k_s k_t)|dt$. The integrals converge provided the integrands are uniformly convergent. By theorem 2.2.7 this follows if $\{k_s\},\{k_t\}$ are both regular since the pointwise product of regular sets is regular. Since the convergence is uniform on regular sets we require a condition to obtain a regular operator.

<u>Definition</u>:

An operator K on R is defined to be <u>regular</u> if the set $\{Kx:\|x\|\le 1\}$ is bounded and regular. That is K maps each bounded subset of R into a bounded and regular set.

It is shown [2, Theorem 3.1] that K maps each bounded subset of R into a regular set if the kernel is (properly) Riemann integrable with respect to (s,t) and also with respect to t for all s. By the above definition such an operator is regular. However this hypothesis may be weakened slightly.

Lemma 2.4.2

Let $k(s,t)$ be (properly) Riemann integrable with respect to t for all s and also with respect to s for all t. Then if K is the integral operator with kernel k, $KR \subset R$, K is bounded and if the set $\{k_t(s):0\le t\le 1\}$ is regular then the operator K is regular.

Proof:

The hypotheses on the kernel $k(s,t)$ imply the existence of the repeated integrals [22, p. 516] and thus $KR \subset R$. These conditions also imply that $k(s,t)$ is bounded and thus

$$\|K\| = \sup_s \int_0^1 |k(s,t)|\,dt \le M \text{ where } M = \sup_{0\le s,t\le 1} |k(s,t)|$$

Hence K is a bounded operator.

From the definition of a regular set, $\{Kx:\|x\|\le 1\}$ is regular if for each $m=1,2,\ldots$ there exist continuous functions $(Kx)^m$, $(Kx)_m$ such that $(Kx)_m \le Kx \le (Kx)^m$ and $\Phi[(Kx)^m-(Kx)_m]\to 0$ as $m\to\infty$, uniformly for $x\in\mathcal{B}$, and the sets $\{(Kx)^m:x\in\mathcal{B}\}$, $\{(Kx)_m:x\in\mathcal{B}\}$ are equicontinuous for each m.

Since $\{k_t\}$ is regular there exist continuous functions $k_t^{\,m}(s)$, $k_{mt}(s)$ such that $k_{mt}(s) \le k_t(s) \le k_t^m(s)$, $\Phi(|k_t^m-k_{mt}|)\to 0$ as $m\to\infty$, uniformly with respect to t, and the sets $\{k_{mt}\}$, $\{k_m^t\}$ are equicontinuous for each m.

Since $x(t)$ is integrable, then so are the non-negative functions $x^+(t)$, $x^-(t)$ where $x(t) = x^+(t)-x^-(t)$. Thus

$$k_m(s,t)x^+(t) \le k(s,t)x^+(t) \le k^m(s,t)x^+(t),$$

$$k_m(s,t)x^-(t) \le k(s,t)x^-(t) \le k^m(s,t)x^-(t) \quad \text{and so}$$

$$k^m(s,t)x^+(t) - k_m(s,t)x^-(t) \ge k(s,t)(x^+(t)-x^-(t)) = k(s,t)x(t) \quad (2.4.1)$$

Hence defining the function

$$(Kx)^m(s) = \int_0^1 k^m(s,t)x^+(t) - k_m(s,t)x^-(t)dt \tag{2.4.2}$$

and $(Kx)_m(s)$ similarly, we obtain for each integrable $x \in \mathcal{B}$, the continuous functions $(Kx)^m$, $(Kx)_m$ such that

$$(Kx)^m(s) \geq (Kx)(s) \geq (Kx)_m(s) \tag{2.4.3}$$

Since the sets $\{k_{mt}\}$, $\{k^m_t\}$ are equicontinuous for each m, the sets $\{\int_0^1 k^m(s,t)x^+(t)dt : 0 \leq x^+ \leq 1\}$ and $\{\int_0^1 k_m(s,t)x^-(t)dt : 0 \leq x^- \leq 1\}$ are also equicontinuous by the argument of lemma 2.3.2. Since the sum of equicontinuous sets is equicontinuous, it is inferred from (2.4.2) that the sets $\{(Kx)^m : \|x\| \leq 1\}$ are equicontinuous for each m. Similarly $\{(Kx)_m : \|x\| \leq 1\}$ are equicontinuous for each m. Finally

$$\begin{aligned}\Phi[(Kx)^m - (Kx)_m] &= \int_0^1 \{\int_0^1 (k^m(s,t)x^+(t) - k_m(s,t)x^-(t)) - (k_m(s,t)x^+(t) - k^m(s,t)x^-(t))dt\}ds \\ &= \int_0^1 \{\int_0^1 (k^m(s,t) - k_m(s,t))|x(t)|dt\}ds\end{aligned}$$

By the definition of the integral operator (2.4.2), it is implicitly assumed (without loss of generality) that the functions $k^m(s,t)$ and $k_m(s,t)$ are (properly) Riemann integrable with respect to t for all s. These functions are defined to be continuous (and hence Riemann integrable) functions of s for all t. Since $x \in R$, these conditions imply the existence and equality of the two repeated integrals as before, and the above integral may be rewritten

$$\int_0^1 \{\int_0^1 (k^m(s,t)-k_m(s,t))ds\} |x(t)| dt \tag{2.4.4}$$

Since $\{k_t\}$ is regular the inner integral converges to zero as $m\to\infty$, uniformly with respect to t. Hence $\Phi[(Kx)^m-(Kx)_m]\to 0$ as $m\to\infty$, uniformly for $x\in\mathcal{B}$, and thus K is a regular operator.

It is noted that, corresponding to the fact that continuity with respect to s and t separately does not imply continuity with respect to (s,t), the hypotheses of integrability with respect to s and t separately do not imply the existence of the double integral (compare for example [22, example 4, p. 519]).

The analogous condition to a collectively equicontinuous family of subsets in the space C would be a collectively regular family of subsets in the space R.

<u>Definition</u>:

The family of sets $\{E_n\}$ in the space R is <u>collectively regular</u> iff the set $\{\bigcup_n E_n\}$ is regular. That is, for each $x_n\in E_n$ and each m=1,2,... there exist $x_{nm}, x_n^m \in C$ such that $x_{nm}\le x_n\le x_n^m$, $\Phi(x_n^m - x_{nm})\to 0$ as $m\to\infty$, uniformly for $x_n\in E_n$ and uniformly in n, and for each fixed m, the sets $\{E_n^m\} = \{x_n^m : x_n\in E_n, n\ge 1\}$ and $\{E_{nm}\} = \{x_{nm} : x_n \in E_n, n\ge 1\}$ are collectively equicontinuous.

<u>Definition</u>:

A set $\{K_n : n\ge 1\}$ of linear operators on the space R is <u>collectively regular</u> iff the family of sets, $\{K_n x : \|x\|\le 1, n\ge 1\}$, is uniformly bounded and collectively regular.

<u>Lemma 2.4.3</u>

Let the integral operator K satisfy the hypotheses of lemma 2.4.2. Let $\{K_n\}$ be the sequence of approximate operators obtained from the

operator K by a sequence of positive quadrature rules. Then the set $\{K_n : n\geq 1\}$ is collectively regular.

Proof:

Since $K_n x(s) = \Phi_n(k_s x) = \Phi(P_n k_s x)$, the hypotheses on the kernel imply $K_n R \subset R$ as in lemma 2.4.2. Since k is bounded, each $K_n \epsilon [R]$. The sequence of positive quadrature rules is pointwise convergent on R and thus the sequence $\{\|\Phi_n\|\}$ is uniformly bounded by the Banach-Steinhaus theorem as in lemma 2.3.3. It is concluded $\|K_n\| = \sup\limits_{\substack{\|x\|\leq 1 \\ 0\leq s\leq 1}} |\Phi_n(k_s x)| \leq B.M$ and $\{K_n\}$ is uniformly bounded.

Applying a positive quadrature rule to obtain the approximate operator, K_n, define the continuous functions (compare (2.4.2)) $(K_n x)^m(s) = \Phi_n(k_s^m x^+ - k_{ms} x^-)$. Since Φ_n is positive, $(K_n x)^m(s) \geq (K_n x)(s)$ by the inequality (2.4.1). $(K_n x)_m(s)$ is defined similarly.

For two arbitrary points s_1, s_2 such that $|s_1 - s_2| < \delta$ $|\Phi_n(k_{s_1}^m x^+ - k_{s_2}^m x^+)| \leq \|\Phi_n\| . \|x^+\| \sup\limits_t \omega(k_t^m, \delta)$. Thus

$$\sup_n \sup_{0\leq x^+ \leq 1} \omega(\Phi_n(k_s^m x^+), \delta) \leq B \sup_t \omega(k_t^m, \delta)$$

Since $\{k_t^m\}$ is equicontinuous for each m, $\sup\limits_t \omega(k_t^m, \delta) \to 0$ as $\delta \to 0$ and thus the set $\{\Phi_n(k_s^m x^+) : 0\leq x^+ \leq 1\}$ is collectively equicontinuous for each m. Similarly $\{\Phi_n(k_{ms} x^-) : 0\leq x^- \leq 1\}$ is collectively equicontinuous for each m since $\{k_{mt}\}$ is equicontinuous for each m. Hence the sets $\{(K_n x)_m(s) : x\epsilon \mathcal{B}, n\geq 1\}$, $\{(K_n x)(s) : x\epsilon \mathcal{B}, n\geq 1\}$ are collectively equicontinuous for each m.

Finally $\Phi[(K_n x)^m - (K_n x)_m] = \int_0^1 \{\Phi_n[(k_s^m x^+ - k_{ms} x^-) - (k_{ms} x^+ - k_s^m x^-)]\} ds$

$$= \int_0^1 \{\Phi_n(k_s^m - k_{ms})|x|\} ds$$

$$= \Phi_n(|x| \int_0^1 k^m(s,t) - k_m(s,t) ds).$$

Thus $|\Phi[(K_n x)^m - (K_n x)_m]| \leq B.\|x\| \sup_t |\int_o^1 k_t^m(s) - k_{mt}(s)ds|$ and the final term $\to 0$ as $m\to\infty$, uniformly for $x\epsilon\mathcal{B}$ and uniformly with respect to n, since $\{k_t\}$ is regular.

Thus $\{K_n x : x\epsilon\mathcal{B}, n\geq 1\}$ is collectively regular and uniformly bounded, and by definition the set of operators $\{K_n : n\geq 1\}$ is collectively regular.

From these results we obtain a theorem for the space R analogous to theorem 2.3.4.

Theorem 2.4.4

Let K be a bounded integral operator on R. Let $\{K_n\}$ be a sequence of approximating operators obtained by a convergent sequence of positive quadrature rules. Then it is sufficient for the application of the convergence theorem (theorem 1.3.2) that the sets $\{k_s, 0\leq s\leq 1\}$, $\{k_t : 0\leq t\leq 1\}$ be regular.

Proof:

1. $\{k_s\}$ regular implies that $K_n \to K$ pointwise (lemma 2.4.1)
2. $\{k_t\}$ regular implies that K is a regular operator (lemma 2.4.2) and $\{K_n : n\geq 1\}$ is a set of collectively regular operators (lemma 2.4.3).

Since integral operators on R converge uniformly on regular sets, the proof and conclusions of theorem 1.3.2 follow.

Thus we have obtained the more "natural" conditions for the applicability of the general theory in the space R; namely that the sets $\{k_s\}$, $\{k_t\}$ are regular. The condition of $\{k_t\}$ regular implies that k(s,t) is a (properly) Riemann integrable function of s for all t by the definition of the Riemann integral. Similarly $\{k_s\}$ regular implies that k is a (properly) Riemann integrable function of t for all s.

The result of Theorem 2.4.4 implies that, in general, we may regard the Fredholm integral equation in the space R, and seek solutions in the space R (compare section 2.1). In the case that the equation is actually in the space C, the solution obtained will automatically be continuous, and if, in fact, the kernel is continuous, then the conditions $\{k_s\}$, $\{k_t\}$ regular reduce to the conditions $\{k_s\}$, $\{k_t\}$ equicontinuous (theorem 2.3.4).

These conditions are more general than those obtained by Anselone in [2]. The condition that $\{k_s\}$ be regular requires the existence of continuous functions $k_s^m(t)$, $k_{ms}(t)$ but does not make any particular requirement on the variable s, other than integrability. The condition of uniform t-integrability requires that these functions also be continuous in s, which leads to the compactness condition: $\Phi(|k_{s_1} - k_{s_2}|) \to 0$ as $s_1 \to s_2$.

Since the regularity conditions do not require that the kernel $k(s,t)$ be continuous in s, or that the discontinuities of $k_s(t)$ lie on a curve which is a continuous function of s, the operator is not necessarily compact [in C]. For example the function

$$k(s,t) = \begin{cases} 0, & s<1/2 \\ 1, & s\leq 1/2 \end{cases}$$

is neither uniformly t-integrable nor the kernel of a compact operator (in Anselone's terminology) but does satisfy the conditions $\{k_s\}$, $\{k_t\}$ regular. Thus this is the kernel of a regular operator to which the general theory is applicable.

It is clear from the definition that a compact integral operator (in C) is a regular operator. Since a uniformly t-integrable kernel implies that the operator is compact [2, theorem 3.2], a uniformly t-integrable kernel also gives a regular operator (this also follows directly from the definition using a construction like that in (2.4.2)).

While the sufficient conditions of theorem 2.4.4 are neat and symmetrical they can in fact be weakened. Although $\{k_t\}$ equicontinuous implies that the operator K is compact (lemma 2.3.2), the compactness of the operator also follows from the much weaker condition $\int_0^1 |k(s,t)-k(s',t)|dt \to 0$ as $s' \to s$, uniformly for $0 \le s,s' \le 1$ [2,5]. Analogously, although $\{k_t\}$ regular implies that K is a regular operator (lemma 2.4.2), this property also follows from weaker conditions.

Lemma 2.4.5

Let the kernel k(s,t) be (properly) Riemann integrable with respect to s for all t and with respect to t for all s. Since $k_s \in R$, there exists k_s^m, $k_{ms} \in C$ such that $k_{ms} \le k_s \le k_s^m$ and

$$\Phi(k_s^m - k_{ms}) \to 0 \text{ as } m \to \infty \text{ for each } s. \tag{2.4.5}$$

Suppose that k_s^m, k_{ms} satisfy the compactness condition

$$\int_0^1 |k^m(s_1,t)-k^m(s_2,t)|dt \to 0 \text{ as } s_1 \to s_2,\ 0 \le s_1, s_2 \le 1, \text{ for each } m. \tag{2.4.6}$$

Then the integral operator K is regular.

Proof:

As in (2.4.2) define $(Kx)^m(s) = \int_0^1 k^m(s,t)x^+(t)-k_m(s,t)x^-(t)dt$ and $(K_x)_m(s)$ similarly. The inequality (2.4.3) follows as in lemma (2.4.2). From [4, proposition 4.1] the compactness condition implies that the convergence in (2.4.6) is uniform and the operators with the kernels k^m, k_m are compact. Thus the functions $(Kx)^m(s)$, $(Kx)_m(s)$ are (uniformly) continuous and in fact the sets $\{(K_x)^m : \|x\| \le 1\}$, $\{(Kx)_m : \|x\| \le 1\}$ are equicontinuous for each m.

Since the function $\int_0^1 k(s,t)dt$ is a continuous function of s on $[0,1]$ by (2.4.6) the functions of (2.4.5) are boundedly convergent. Hence $\Phi[(Kx)^m - (Kx)_m] \leq \|x\| \cdot \int_0^1 \int_0^1 (k^m(s,t) - k_m(s,t))dt\, ds \to 0$ as $m \to \infty$, uniformly for $x \in \mathcal{B}$. Thus K is a regular operator.

Definition: [10]

A sequence $\{E_n\}$ of sets of functions in R is asymptotically equicontinuous if $\sup_{x \in E_n} |x(s)-x(t)| \to 0$ as $s \to t$ and $n \to \infty$, uniformly for $t \in [0,1]$.

This is a rather weaker condition than collectively equicontinuous defined above. Similarly we may define a weaker form than collectively regular.

Definition:

A sequence $\{E_n\}$ of sets of functions in R is asymptotically regular if for each fixed m, the sets $\{E_n^m\}$, $\{E_{nm}\}$ (see the definition of collectively regular) are asymptotically equicontinuous and $\Phi(x_n^m - x_{nm}) \to 0$ as $m \to \infty$ and $n \to \infty$.

Lemma 2.4.6

Let $\{E_n\}$ be asymptotically regular and each E_n a regular set in R. Then $\{E_n\}$ is collectively regular.

Proof:

Since each E_n is regular the sets E_n^m, E_{nm} are equicontinuous for each m,n. Further $\Phi(x_n^m - x_{nm}) \to 0$ as $m \to \infty$, uniformly with respect to $x_n \in E_n$ for each n. Since $\{E_n\}$ is asymptotically regular the sets $\{E_n^m\}$, $\{E_n^m\}$ are asymptotically equicontinuous. Anselone's result using a construction based on the usual proof of the Arzela-Ascoli theorem [5, proposition 2.4] implies that $\bigcup_n E_n^m$ is equicontinuous. That is, in the present terminology, $\{E_n^m\}$, $\{E_{nm}\}$ are collectively equicontinuous for each m.

Further $\Phi(x_m^n - x_{nm}) \to 0$ as $m\to\infty$ and $n\to\infty$. Thus, given $\epsilon>0$, there exists M_1, N such that $\Phi(x_m^n - x_{nm})<\epsilon$ for $n>N$, $m>M_1$. Since this convergence also holds for each n, we also have for $n\leq N$, $\Phi(x_m^n - x_{nm})<\epsilon$ for $m>M(n)$. Thus $\Phi(x_m^n - x_{nm})<\epsilon$ for $m>M$ and for all n where $M = \max(\max_{n\leq N} M(n), M_1)$, i.e., $\Phi(x_m^n - x_{nm})\to 0$ as $m\to\infty$, uniformly with respect to n. Hence $\{E_n\}$ is collectively regular by definition.

Lemma 2.4.7

Let the kernel k(s,t) satisfy the hypotheses of lemma 2.4.5 and also suppose that $\{k_s\}$ is regular. Then $\{K_n : n\geq 1\}$ is collectively regular where $\{K_n\}$ is the sequence of approximate operators obtained from K by a sequence of positive quadrature rules.

Proof:

Defining the functions $(K_n x)^m(s)$, $(K_n x)_m(s)$ analogously to those in lemma 2.4.5 we have

$$|(K_n x)^m(s) - (K_n x)^m(s')| \leq \Phi_n(|k_s^m - k_{s'}^m|)\|x^+\| + \Phi_n(|k_{ms} - k_{ms'}|)\|x^-\|.$$

Since $\{k_s\}$ is regular the sets $\{k_s^m\}$, $\{k_{ms}\}$ are equicontinuous, and hence $\Phi_n(|k_s^m - k_{s'}^m|) \to \Phi(|k_s^m - k_{s'}^m|)$ as $n\to\infty$, uniformly for $0\leq s,s'\leq 1$ by lemma 2.4.1. By (2.4.6), $\Phi(|k_s^m - k_{s'}^m|)\to 0$ as $s'\to s$, uniformly for $0\leq s\leq 1$ (compare lemma 2.4.5).

Thus $(K_n x)^m(s) - (K_n x)^m(s')\to 0$ as $n\to\infty$ and $s'\to s$, uniformly for $\|x\|\leq 1$ and $0\leq s\leq 1$. This implies that the sequence of sets $\{(K_n x)^m : \|x\|\leq 1\}$ is asymptotically equicontinuous for each fixed m.

Further $\Phi[(K_n x)^m(s) - (K_n x)_m(s)] = \int_0^1 (\Phi_n(k_s^m - k_{ms})|x|)ds$. As $n\to\infty$, the right side converges uniformly with respect to s to $\int_0^1\int_0^1 (k^m(s,t) - k_m(s,t))|x|\,dt\,ds$ by the same argument as above. This expression

converges to zero as $m\to\infty$ using (2.4.5) (compare lemma 2.4.5).

Thus the sequence of sets $\{K_n x: \|x\| \le 1\}$ is asymptotically regular. Since each of the sets $\{K_n x: \|x\| \le 1\}$ has a finite dimensional range, it is regular. Thus by lemma 2.4.6 the sets $\{K_n x: \|x\| \le 1, n \ge 1\}$ are collectively regular. Uniform boundedness follows as in lemma 2.4.3 and hence $\{K_n : n \ge 1\}$ is collectively regular.

In connection with this proof, note that, as defined, the sets $\{(K_n x)^m(s): x \in \mathcal{B}\}$ are not equicontinuous for each m,n. This may be compared to Anselone's treatment of discontinuous kernels [5] where $KR \subset C$ but $K_n R \not\subset C$ in general. However since $k_t(s) \in R$, there exist $k_t^{\ell}(s)$, $k_{\ell t}(s) \in C$ such that $k_{\ell t} \le k_t \le k_t^{\ell}$ and $\Phi(k_t^{\ell} - k_{\ell t}) \to 0$ as $\ell \to \infty$ for each t. Thus $\{\Phi_n(k_s^{\ell} x^+): 0 \le x^+ \le 1\}$ is equicontinuous for each ℓ and each n since it involves a finite number of continuous functions. Hence $\{(K_n x)^{\ell}(s): x \in \mathcal{B}\}$ (defined in terms of the functions k^{ℓ}, k_{ℓ}) is equicontinuous for each ℓ and each n.

Hence in order to see that the lemma 2.4.6 is in fact applicable, we may take for the majorizing and minorizing sets of functions $(K_n x)^p$, $(K_n x)_p$ where $p=m$ for $n>N$ and $p=\ell$ for $n \le N$. For $p=m$ the collection is asymptotically equicontinuous. Thus given $\epsilon>0$, there exist $\delta(\epsilon)$, $N(\epsilon)$ such that

$$\sup_{n>N(\epsilon)} \ \sup_{\substack{x \in \mathcal{B} \\ |s-s'|<\delta(\epsilon)}} \left|(K_n x)^p(s) - (K_n x)^p(s')\right| < \epsilon$$

For $n \le N(\epsilon)$ we have a finite collection of equicontinuous sets with $p=\ell$ and thus

$$\sup_{n \le N(\epsilon)} \ \sup_{\substack{x \in \mathcal{B} \\ |s-s'|<\delta(t)}} \left|(K_n x)^p(s) - (K_n x)^p(s')\right| < \epsilon$$

Combining these two collections of sets of functions it is clear that $\{(K_n x)^p : x\epsilon\mathcal{B}\}$ is collectively equicontinuous for each p.

For $n\le N(\varepsilon)$, $\int_0^1 (\Phi_n(k_s^p - k_{ps})|x|)ds = \Phi_n[\int_0^1 k_t^{\ell} - k_{\ell t} ds]|x|$ which converges to zero as $\ell\to\infty$ for each n since the inner integral converges for each t. For $n>N$ the expression converges as in the lemma 2.4.7 and hence $\Phi[(K_n x)^p - (K_n x)_p]\to 0$ as $p\to\infty$ uniformly with respect to n, as required in the proof of lemma 2.4.6.

Collecting these results we obtain the more general convergence theorem

Theorem 2.4.8

Let the kernel k(s,t) satisfy $k_t \epsilon R$ and $\{k_s\}$ regular. Suppose further that the majorizing and minorizing functions k_s^m, $k_{ms} \epsilon C$ satisfy the condition $\int_0^1 |\ell(s,t)-\ell(s',t)|dt\to 0$ as $s'\to s$, $0\le s,s'\le 1$, for each m.

Then $K_n\to K$, K is regular and $\{K_n\}$ is collectively regular. Thus the convergence theorem (theorem 1.3.2) follows.

Proof:

Since $\{k_s\}$ is regular, pointwise convergence of the approximate operators follows by lemma 2.4.1. K regular and $\{K_n\}$ collectively regular follow from lemmas 2.4.5 and 2.4.7 respectively. Since integral operators on R converge uniformly on regular sets, the proof and conclusions of theorem 1.3.2 follow as in theorem 2.4.4.

2.5 Quadrature for Unbounded Functions

Since we are ultimately concerned with kernels with an unbounded singularity, a next step in progressively relaxing the conditions on the kernel is to consider the case where the kernel is absolutely integrable.

This leads to an investigation of numerical quadrature for integrable singular functions. As in section 1.2, this problem may be considered either by approximate product integration or by a direct method.

The direct method is the process of "ignoring the singularity" which was studied by Davis and Rabinowitz [13,40]. That is, if $x(t)$ becomes singular at a point ξ, then one defines $x(\xi) = 0$ (or any other finite value) and then approximates the integral by a usual numerical quadrature rule. It is shown that the process may or may not be valid depending on the nature and position of the singularity. Thus if the singular point, ξ, is a "rational" point in the range of integration then compound quadrature rules do approximate the integral when the integrand is monotone near ξ. On the other hand if $x(t)$ is monotonic but ξ is an irrational point then [13, theorem 5] the process may not be theoretically valid and in general only "lim inf convergence" can be asserted. These results were generalized by Miller [32] who replaced the assumption of monotonicity by the more general condition that $x(t)$ can be dominated by a monotone, integrable function. However the restriction to rational singularities is retained.

Consider the weakly singular operator. The integrand in this case has the form $k(s,t)x(t)/|t-s|^{\alpha}$, $0<\alpha<1$. Regarding s as temporarily fixed, the above remarks imply that convergence for an irrational value of s cannot be asserted for a sequence of approximations obtained by a direct method. Thus if K_n denotes the approximate operator obtained by a quadrature of this type, we certainly cannot assert $K_n x(s) \rightarrow Kx(s)$ uniformly in s. That is, we cannot assert pointwise convergence, and the first hypothesis of the convergence theorem 1.3.2 is not satisfied.

However, the method of approximate product integration neatly solves this problem. The singular part of the integrand is absorbed into

the integration operator. Let the kernel $k(s,t) = p(s,t)\ell(s,t)$, where $p(s,t)$ ($p(s,t) = |t-s|^{-\alpha}$, $0<\alpha<1$) is absolutely integrable and $\ell(s,t)$ is continuous.

Lemma 2.5.1

The sequence $\{K_n\}$ of approximate operators obtained by approximate product integration converges pointwise to the weakly singular operator, K, if the partial modulus of continuity, $\omega_t(\delta)$, of the continuous part, $\ell(s,t)$, of the kernel exists as a simple modulus of continuity.

Proof:

Using corollary 2.2.5

$$\|K_n x - Kx\| = \sup_s |\Phi[p_s\{P_n\ell_s x - \ell_s x\}]| \leq C \sup_s\{|\Phi(p_s)|\,\omega(\ell_s x,\delta)\}$$

$$\leq C.\|Q\|[\|x\|\omega_t(\delta) + L\ \omega(x,\delta)]\text{, by (2.3.1),}$$

where C is a constant depending on the fixed degree of the interpolating polynomial, Q is the weakly singular integral operator with kernel p, and L is the absolute bound for $\ell(s,t)$. If $\{\ell_s\}$ is equicontinuous we have convergence for each fixed $x \in C$.

Note that in practical cases part or all of the function $\ell(s,t)$ may be absorbed into the integral operator in order to obtain better convergence properties of the polynomial approximations. Also the Gauss type formulas may be applied using Theorem 2.2.6.

Lemma 2.5.2

The integral operator K is compact if $\{\ell_t\}$ is equicontinuous and

$$\int_0^1 |p(s,t)-p(s',t)|\,dt \to 0 \text{ as } s' \to s, \qquad 0 \leq s,s' \leq 1 \tag{2.5.1}$$

Proof:

The convergence in (2.5.1) is uniform and $\max_s \Phi(|p_s|)$ exists [4, Theorem 4.1]. Thus K is bounded and $\|K\| \le L.\max_s \Phi(|p_s|) = L.\|Q\|$. For $|s_1 - s_2| < \delta$,

$$|Kx(s_1)-Kx(s_2)| \le \int_0^1 |p(s_1,t)\ell(s_1,t)-p(s_2,t)\ell(s_2,t)|.|x(t)|dt$$

$$\le \|x\|\int_0^1 (|p(s_1,t)|\omega(\ell_t,\delta)+|p(s_1,t)-p(s_2,t)|.|\ell(s_2,t)|)dt$$

$$\le \|x\|(\omega_s(\delta).\|Q\| + L.\int_0^1 |p(s_1,t)-p(s_2,t)|dt)$$

Thus $K\mathcal{B}$ is bounded and equicontinuous using (2.5.1) and $\{\ell_t\}$ equicontinuous.

The lemma was expressed in this form to show the analogy with the previous lemmas, but it is clear that if $\{\ell_t\}$ is not equicontinuous, the operator will still be compact if k satisfies (2.5.1).

Lemma 2.5.3

The sequence $\{K_n\}$ of approximate operators is collectively compact if (2.5.1) holds and $\ell(s,t)$ is continuous.

Proof:

By corollary 2.2.5, the sequence $\{Q_n\}$ of approximate quadrature rules is pointwise convergent and the Banach-Steinhaus theorem implies that the sequence is uniformly bounded as in lemma 2.3.3. Thus

$$\|K_n x\| = \sup_s |Q_n(\ell_s x)| \le \|Q_n\|.\|x\|.\sup_s\|\ell_s\| \le B.L.\|x\|$$

Hence $\{K_n\mathcal{B}: n\ge 1\}$ is uniformly bounded.

For two arbitrary points s_1,s_2 such that $|s_1-s_2|<\delta$,

$$|(K_nx)(s_1)-(K_nx)(s_2)| = |\int_0^1 p(s_1,t)P_n(\ell(s_1,t)x(t))-p(s_2,t)P_n(\ell(s_2,t)x(t))dt|$$
$$\leq \|x\|[\int_0^1 |p(s_1,t)-p(s_2,t)|.P_n(|\ell(s_1,t)|)dt$$
$$+\int_0^1 |p(s_2,t)|P_n(|\ell(s_1,t)-\ell(s_2,t)|)dt]$$

Since $\{\ell_s\}$ is equicontinuous the integrals converge uniformly with respect to s (by lemma 2.5.1) to

$$[\int_0^1 |p(s_1,t)-p(s_2,t)|.|\ell(s_1,t)|dt + \int_0^1 |p(s_2,t)|.|\ell(s_1,t)-\ell(s_2,t)|dt]$$

This term converges to zero as $s_1 \to s_2$ by lemma 2.5.2 and hence $(K_nx)(s_1)-(K_nx)(s_2) \to 0$ as $n\to$ and $s_1 \to s_2$, uniformly for $\|x\| \leq 1$ and $0 \leq s_2 \leq 1$. That is $\{K_n\mathcal{B}\}$ is asymptotically equicontinuous. Since each K_n has a finite dimensional range it is equicontinuous, and hence $\{K_n\mathcal{B}:n\geq 1\}$ is collectively equicontinuous.

It follows that $\{K_n:n\geq 1\}$ is collectively compact.

Note that if the approximate product integretion rule is obtained by piecewise polynomial interpolation, the collective equicontinuity is easily obtained by

$$|K_nx(s_1)-K_nx(s_2)| = |\int_0^1 p(s_1,t)P_n^m\ell(s_1,t)x(t)-p(s_2,t)P_n^m\ell(s_2,t)x(t)dt|$$
$$\leq \int_0^1 |p(s_1,t)-p(s_2,t)||P_n^m\ell(s_1,t)x(t)|dt$$
$$+\int_0^1 |p(s_2,t)|.|P_n^m(\ell(s_1,t)-\ell(s_2,t))x(t)|dt$$
$$\leq \|x\|[L.\|L_m\|\int_0^1 |p(s_1,t)-p(s_2,t)|dt + \|Q\|.\|L_m\|\omega_s(\delta)]$$

which gives the result as in lemma 2.5.2.

Hence we obtain Atkinson's result [7] for weakly singular kernels in terms of the current development.

Theorem 2.5.4

Let K be an integral operator such that the kernel $k(s,t) = p(s,t)\ell(s,t)$ satisfies $p_s \in L^1(0,1)$ for $0 \le s \le 1$ and $\int_0^1 |p(s_1,t)-p(s_2,t)|dt \to 0$ as $s_1 \to s_2$, $0 \le s_1, s_2 \le 1$. If $\{K_n\}$ is a sequence of approximating operators obtained by applying a sequence of approximate product integration rules, then it is sufficient for the application of the collectively compact operator theory that the two partial moduli of continuity of $\ell(s,t)$ exist as simple moduli of continuity.

Proof:

The theorem follows from lemmas 2.5.1, 2.5.2, 2.5.3 as in Theorem 2.3.4.

The remarks in this section suggest that the approximation of a strongly singular integral operator is probably best achieved by some method of approximate product integration.

3. APPLICATION TO SINGULAR INTEGRAL EQUATIONS

3.1 Preliminary Remarks

It has been shown in the preceding chapter that the conditions for the application of the abstract convergence theorem are the pointwise convergence of the approximate operators and the identification of those sets over which this convergence is uniform. These conditions were ultimately reduced to the requirement that the kernel of the integral operator have suitable properties.

The application is relatively straightforward for continuous kernels. In the case of a weakly singular kernel the convergence properties of approximate quadrature formulas for improper integrals cause problems. These are avoided by approximation with an approximate product integration rule (section 2.5). However, this technique cannot be immediately extended to singular equations with a Cauchy kernel since this kernel is not absolutely integrable. Further the integral must be interpreted as a Cauchy principal value and hence conditions for the existence of this integral as well as the convergence of a sequence of principal value integrals must be considered. Stewart [42] has observed that even if a sequence of functions is uniformly convergent in C, and the Cauchy integrals of all functions in the sequence exist, the sequence of Cauchy integrals may not converge.

Hence the properties of the singular integral operator are considered in order to determine conditions analogous to those in Chapter 2, for the pointwise and uniform convergence of a sequence of approximations to such operators.

It is assumed that the definition and properties of a function satisfying a Hölder condition with index μ (denoted $\varphi \in H(\mu)$) are known (for example [33, chapter 1]). Functions which satisfy the Hölder condition with larger

exponents are "more benign" than those which satisfy the condition with a smaller exponent (that is $\alpha<\beta$ implies $H(\beta) \subset H(\alpha)$, [34, chapter 4]). The statements $\varphi \in H(\mu)$ and $\omega(\varphi,\delta) \leq M\delta^{\alpha}$ are equivalent for an arbitrary constant M. The class of functions satisfying the $H(\mu)$ condition form a Banach space with the appropriate norm (Lorentz [29, p. 50]).

The Riemann integral (with respect to t) of the function $\varphi(t)/(t-s)$ may not exist even as an improper integral if $\varphi \in C$, but (Mikhlin [31, p. 115]) it does exist in the sense of the Cauchy principal value if $\varphi \in H(\mu)$, $0<\mu\leq 1$. The idea of a principal value is easily extended to contour integrals (Muskhelishvili [33], Pogorzelski [38]) and we note the important Privalov theorem:

<u>Theorem 3.1.1</u> (The Privalov Theorem)[38, p. 439], [33, p. 46]

If a complex function $\varphi(t)$, defined on a smooth arc L, satisfies the $H(\mu)$ condition, then the values of the Cauchy singular integral, $\Phi(s) = \int_L \frac{\varphi(t)}{t-s}dt$, at points s on the line of integration, except in arbitrary small neighborhoods of those ends where $\varphi(t)\neq 0$, satisfy the $H(\mu)$ condition for $\mu<1$, and the $H(1-\epsilon)$ condition for $\mu=1$, where ϵ is an arbitrary small positive number.

We note also the Poincaré-Bertrand transformation formula for iterated singular integrals [33, p. 57]: If L is a smooth arc or contour, $\varphi(t,u)$ a function of the two points t,u on this line, satisfying a Hölder condition with respect to t and u, and let s be a fixed point on L not coinciding with one of its ends, then

$$\int_L \frac{dt}{t-s} \int_L \frac{\varphi(t,u)}{u-t}du = -\pi^2\varphi(s,s) + \int_L \int_L \frac{\varphi(t,u)}{(t-s)(u-t)}\, dt\, du \ .$$

3.2 Properties of the Singular Integral Operator

The Hilbert transformation of a function f(t) is defined by

$$g(s) = Hf(s) = H_s[f(t)] = \frac{1}{\pi} P \int \frac{f(t)}{t-s} dt.$$

In the terminology of Fourier series this is often called the conjugate function to f(t). Since the Cauchy principal value integral is conditionally but not absolutely convergent, the existence of g(s) is not trivial even if f(t) is continuous. Zygmund [49] notes (by means of the conjugate Fourier series: $\sum_{n=2}^{\infty} \frac{\sin nx}{n \log n}$, $\sum_{n=2}^{\infty} \frac{\cos nx}{n \log n}$) that the function conjugate to a continuous function need not be continuous or even bounded. Thus for the Hilbert operator, H, it cannot even be asserted that $HC \subset R$, in contrast to Anselone's requirement that $KR \subset C$ for a mildly discontinuous kernel.

However the Privalov theorem shows that the linear operator H transforms $H(\mu)$ into itself for $0<\mu<1$. Mikhlin [31], Muskhelishvili [33] and Pogorzelski [38] assume the known functions are Hölder continuous and derive solutions of the singular integral equation in the space of Hölder continuous functions. Thus the apparent natural analogy for the space C in the case of the Fredholm integral equations, is the Banach space $H(\mu)$ in the case of singular integral equations.

The function $f \in H(\mu)$ will be said to be of exact order μ if it does not belong to any smaller space, i.e., $f(t) \notin H(\nu)$, $\nu > \mu$. Then we have the minor extension to the Privalov theorem:

Lemma 3.2.1

Let $f(t) \in H(\mu)$ be of exact order μ, $0<\mu<1$. Then the Hilbert transformation of f is also of exact order μ.

Proof:

Let $g(s) = H_s[f(t)]$. Suppose $g(s) \epsilon H(\nu)$, $\nu > \mu$. (This does not contradict the Privalov theorem since $H(\nu) \subset H(\mu)$). Then

$$H_t[g(s)] = -f(t) + \frac{1}{\pi^2} P\int_L f(u)du \; P\int \frac{ds}{(s-t)(u-s)}$$

But as is easily verified

$$P\int_L \frac{ds}{(s-t)(u-s)} = \frac{1}{u-t} P\int_L \{\frac{1}{s-t} - \frac{1}{s-u}\} ds = 0$$

Thus $H_t[g(s)] = -f(t)$, and the Privalov theorem implies that since $g \epsilon H(\nu)$ then $f \epsilon H(\nu)$, contradicting the hypothesis that f is of exact order μ.

These remarks assume the existence of functions belonging exactly to a given Hölder class. Such functions have been identified, for example by Loud [30] who constructs (based on Knopp's construction of a continuous non-differentiable function) a class of functions satisfying at every point a Hölder condition of prescribed order. Also by Hardy [20] who effectively obtains a similar result [20, p. 303] using Weirstrass' non-differentiable function.

3.3 Convergence in a Hölder Space

Let $H(\beta)$ be the class of functions defined on an interval of length and satisfying a Hölder condition of order β. Let the norm be defined $||x||_\beta = M_1[x] + M_2[s;\beta]$, $x \epsilon H(\beta)$ where

$$M_1[x] = \max_t |x(t)|, \; M_2[x;\beta] = \sup_{0<|t_1-t_2|<\ell} \frac{|x(t_1)-x(t_2)|}{|t_1-t_2|^\beta} .$$

Then the space $H(\beta)$ is a Banach space [29, p. 50].

By an obvious extension of the idea of equicontinuity, we define a set, E, of functions to be equi-Hölder continuous.

Definition:

The set, E, of functions is equi-Hölder continuous of order β [equi-H(β)] if given $\epsilon>0$, there exists $\delta>0$ such that, if $|t_1-t_2|<\delta$, then $\frac{|x(t_1)-x(t_2)|}{|t_1-t_2|^\beta}<\epsilon$ for all $x\epsilon E$ and t_1,t_2 in the interval. Equivalently, in terms of the modulus of continuity, if $\sup_{x\epsilon E}\frac{\omega(x,\delta)}{\delta^\beta}\to 0$ as $\delta\to 0$.

Since the abstract approximation theory implies uniform convergence on compact sets, it is useful to identify the compact sets in H(β). We need also to define the class h(β) (apparently introduced by Hardy [21] as the class Lip$^*\beta$).

Definition: [21]

The function x(t) belongs to the class h(β) if $x(t+h)-x(t) = O(|h|^\beta)$ as $h\to 0$, uniformly in t.

Clearly h(β) is a closed subspace of H(β). Johnson [24] characterizes the relatively compact subsets of sets of functions defined on a compact interval by

Lemma 3.3.1 [24, p. 154]

Let E be a subset of h(1). Then E is relatively compact iff E is bounded and $\lim_{|s-t|\to 0}\frac{|x(s)-x(t)|}{|s-t|}=0$ uniformly on E.

Corollary 3.3.2

If $0<\alpha<\beta\le 1$ then the unit ball of H(β) is compact in $h(\alpha)\subset H(\alpha)$.

Thus the relatively compact subsets of h(α) are identified as bounded and equi-H(α) sets. These sets are also compact in the larger space H(α) (with the same norm). Since the unit ball of H(β) is compact in the topology of H(α), it is a fortiori compact in any coarser topology, and in particular

in the sup-norm topology of $h(0)$ (the space C). Thus the Arzela-Ascoli theorem is reclaimed as a special case.

It was noted that $x \in H(\beta)$ iff the expression $\omega(x,\delta)/\delta^{\beta}$ is bounded for all sufficiently small values of δ. We assert that $x(t)$ has exact order β iff there exists at least one point, t_o, such that x has exact order β at that point. Thus modifying Loud's definition [30] slightly we obtain:

Definition:

A function $x(t)$ satisfies a Hölder condition of exact order β if there exists positive constants K_1, K_2 such that

(a) For any t and any sufficiently small $\delta>0, |x(t+\delta)-x(t)|<K_1\delta^{\beta}$

(b) $\exists\ t_o$ and an infinite sequence, $\{\delta_m\}$, such that $\delta_m \to 0$ as $m\to\infty$ and $|x(t_o+\delta_m)-x(t_o)|>K_2\delta_m^{\beta}$.

In section 2.2 it was seen that convergence of the approximate quadrature rules on C depends on the convergence of the piecewise polynomial approximations of the integrand in C. Thus we are also concerned with the convergence in $H(\beta)$ of the polynomial approximations to a function in a Hölder space.

Lemma 3.3.3

Let L_m denote the Lagrange interpolation operator of degree m on equally spaced intervals, and let P_n^m denote the piecewise, m-th degree polynomial operator over n subintervals of length mh. Then

$$\|x-P_n^m x\|_{\beta} \le K(m) \sup_{0<\delta\le h} \frac{\omega(x,\delta)}{\delta^{\beta}}$$

where $K(m)$ is a constant depending only on m.

Proof:

By lemma 2.2.2

$$M_1[x-P_n^m x] \le m\|L_m\|\omega(x,h) \tag{3.3.1}$$

In order to bound the Hölder constant, let $\rho x = x-P_n^m x$, and for two arbitrary points s_1, s_2 consider the ratio $|\rho x(s_1)-\rho x(s_2)|/|s_1-s_2|^\beta$. If $h \le |s_1-s_2|$ then, using lemma 2.2.2

$$\frac{|\rho x(s_1)-\rho x(s_2)|}{|s_1-s_2|^\beta} \le \sup_t \frac{2|\rho x(t)|}{|s_1-s_2|^\beta} \le 2m\|L_m\|\frac{\omega(x,h)}{h^\beta} \tag{3.3.2}$$

If $|s_1-s_2|<h$ and these two points lie in the same subinterval of length mh, then

$$\frac{|\rho x(s_1)-\rho x(s_2)|}{|s_1-s_2|^\beta} \le |s_1-s_2|^{-\beta}\{|x(s_1)-x(s_2)|+|L_m x(s_1)-L_m x(s_2)|\} \tag{3.3.3}$$

Now $|L_m x(s_1)-L_m x(s_2)| = |\sum_{k=0}^m x(t_k)(\ell_k(s_1)-\ell_k(s_2))|$

$$= |\sum_{k=0}^m (x(t_k)-x(t))(\ell_k(s_1)-\ell_k(s_2))|$$

for an arbitrary fixed point t in the interval since $\sum_{k=0}^m \ell_k(t) = 1$.

$$\le \omega(x,mh) \sum_{k=0}^m |\ell_k(s_1)-\ell_k(s_2)|$$

Further $\ell_k(t)$ is a polynomial of degree m and this is differentiable. Thus $|\ell_k(s_1)-\ell_k(s_2)| \le \max_t|\ell_k'(t)|\cdot|s_1-s_2|$. A bound for this derivative can be crudely estimated from the expression for $\ell_k(t)$ in lemma 2.2.2. It is the sum of m terms whose numerator is bounded by $m!(1+1+\frac{1}{2}+\ldots+\frac{1}{k-1}+\frac{1}{k+1}+\ldots+\frac{1}{m})h^{m-1}$ The denominator is $k!(m-k)!h^m$. Thus

$$\sum_{k=0}^{m} |\ell_k'(t)| \le \frac{1}{h} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!}(1+1+\log m) \le \frac{2^m(2+\log m)}{h}$$

Hence from (3.3.3) we conclude

$$\frac{|\rho x(s_1)-\rho x(s_2)|}{|s_1-s_2|^\beta} \le \frac{\omega(x,|s_1-s_2|)}{|s_1-s_2|^\beta} + \omega(x,mh).2^m(2+\log m).\frac{|s_1-s_2|^{1-\beta}}{h}$$

$$\le \frac{\omega(x,|s_1-s_2|)}{|s_1-s_2|^\beta} + m.2^m.(2+\log m)\frac{\omega(x,h)}{h^\beta} \qquad (3.3.4)$$

If $|s_1-s_2|<h$ but the points lie in different intervals they must of course lie in adjacent intervals. Let L_m be the operator in one interval with basis polynomials ℓ_k and nodes t_k, and L_m^* be the operator in the adjacent interval with basis polynomials ℓ_k^* and nodes t_k'. Suppose $t_{m-1}<s_1<t_m = t_0'<s_2<t_1'$. Then

$$|L_m x(s_1) - L_m^* x(s_2)| = |\sum_{k=0}^{m} x(t_k)\ell_k(s_1)-x(t_k')\ell_k^*(s_2)|$$

$$= |\sum_{k=0}^{m} (x(t_k)-x(t_m))\ell_k(s_1)-(x(t_k')-x(t_o'))\ell_k^*(s_2)|$$

$$\le \omega(x,mh)(\sum_{k=0}^{m-1} |\ell_k(s_1)| + \sum_{k=1}^{m} |\ell_k^*(s_2)|)$$

$$= \omega(x,mh)\sum_{k=0}^{m-1} (|\ell_k(s_1)| + |\ell_{k+1}^*(s_2)|) \qquad (3.3.5)$$

$$\text{Now} |\ell_k(s_1)|+|\ell_k^*(s_2)| = \left|\frac{(s_1-t_0)\dots(s_1-t_{k-1})(s_1-t_{k+1})\dots(s_1-t_m)}{(t_k-t_0)\dots(t_k-t_{k-1})(t_k-t_{k+1})\dots(t_k-t_m)}\right| +$$

$$+ \left|\frac{(s_2-t_0')\dots(s_2-t_k')(s_2-t_{k+2}')\dots(s_2-t_m')}{(t_{k+1}'-t_0')\dots(t_{k+1}'-t_k')(t_{k+1}'-t_{k+2}')\dots(t_{k+1}'-t_m')}\right|$$

$$\le \frac{\frac{m!}{m-k}h^{m-1}(t_m-s_1)}{k!(m-k)!h^m} + \frac{(s_2-t_0')\frac{m!}{k+1}h^{m-1}}{(k+1)!(m-(k+1))!h^m} \text{ using the properties of } s_1,s_2$$

$$= \frac{m!}{k!(m-k-1)!h}\left[\frac{(t_m-s_1)}{(m-k)^2} + \frac{(s_2-t_0')}{(k+1)^2}\right]$$

But $m-k \geq 1$ and $k+1 \geq 1$ for the range of k in (3.3.5) and thus

$$\leq \frac{m!}{k!(m-k-1)!} \cdot \frac{s_2-s_1}{h} = mC_k^{m-1} \cdot \frac{s_2-s_1}{h}$$

Thus since $\sum_{k=0}^{m-1} C_k^{m-1} = 2^{m-1}$, and substituting this result in (3.3.5) we obtain

$$\frac{|\rho x(s_1)-\rho x(s_2)|}{|s_1-s_2|^\beta} \leq \frac{|x(s_1)-x(s_2)|}{|s_1-s_2|^\beta} + \frac{|L_m x(s_1)-L_m^* x(s_2)|}{|s_1-s_2|^\beta}$$

$$\leq \frac{\omega(x,|s_1-s_2|)}{|s_1-s_2|^\beta} + m\ \omega(x,h).m2^{m-1}\ \frac{|s_1-s_2|^{1-\beta}}{h}$$

$$\leq \frac{\omega(x,|s_1-s_2|)}{|s_1-s_2|^\beta} + m^2 2^{m-1}\ \frac{\omega(x,h)}{h^\beta} \qquad (3.3.6)$$

Thus collecting the results of (3.3.2), (3.3.4), (3.3.6) we obtain the estimate

$$M_2[x-P_n^m x;\beta] = \sup_{0<|s_1-s_2|} \frac{|\rho x(s_1)-\rho x(s_2)|}{|s_1-s_2|^\beta}$$

$$\leq K_1(m) \sup_{0<\delta\leq h} \frac{w(x,\delta)}{\delta^\beta} \qquad (3.3.7)$$

where $K_1(m) = \max[2m\|L_m\|, (1+\max\{2(2+\log m), m\} m2^{m-1})]$

Although this estimate may be able to be improved (for example more careful analysis of the case m=1 yields a coefficient of 1 for $\omega(x,h)/h^\beta$ in

(3.3.4), and hence $K_1(1)=2$), it is sufficient for our purposes.

Finally by (3.3.1) and (3.3.7), and assuming $h\le 1$

$$\|x-P_n^m x\|_\beta = M_1[x-P_n^m x] + M_2[x-P_n^m x;\beta]$$

$$\le (m\|L_m\| + K_1(m)) \sup_{0<\delta\le h} \frac{\omega(x,\delta)}{\delta^\beta} = K(m) \sup_{0<\delta\le h} \frac{\omega(x,\delta)}{\delta^\beta}$$

Corollary 3.3.4

If $P_n^m x(t)$ is the piecewise polynomial approximation to $x\epsilon H(\alpha)$ then $\|x-P_n^m x\|_\beta \le K(m).M_2[x;\alpha]\, h^{\alpha-\beta}$, $0<\beta\le\alpha\le 1$.

Proof:

If $x\epsilon H(\alpha)$ then $\omega(x,\delta) \le M_2[x;\alpha]\,\delta^\alpha$. Thus we have $\omega(x,\delta)/\delta^\beta \le M_2[x;\alpha]\,\delta^{\alpha-\beta}$ and the result follows by the theorem.

From the theorem it is concluded that the piecewise polynomial $P_n^m x(t)$ converges to its generating element on the subset $h(\beta) \subset H(\beta)$. It follows from the corollary that this subset includes functions belonging to any space $H(\alpha)$, $0<\beta<\alpha\le 1$. Further it follows easily from the theorem that the convergence is uniform over any equi-$H(\beta)$ set. Thus the convergence is uniform over a bounded set in $H(\alpha)$.

However the result is inconclusive with regard to a function belonging to the space $H(\beta)$. This contrasts to lemma 2.2.2 for the space C where $P_n^m x$ converges to x for any $x\epsilon C$. That the result for the Hölder space cannot be improved in general may be seen by considering a function belonging to $H(\beta)$ but not to any smaller space. It is sufficient to consider the case m=1 of polygonal approximations.

Lemma 3.3.5

If $x\epsilon H(\beta)$ is of exact order β then its polygonal approximation,

$P_n x$, does not converge to x in the norm of the space $H(\beta)$, i.e.,

$\|P_n x - x\|_\beta \not\to 0$ as $n\to\infty$

Proof:

Let $\rho_n x = x - P_n x$. We use the notation of the definition above that $x(t)$ satisfies a Hölder condition of exact order β. Clearly $K_2 < K_1$ and since $K_2 > 0$, there exists $\varepsilon > 0$ such that $(K_2-\varepsilon)/K_1 < 1$. $\{\delta_m\}$ is the infinite sequence occurring in the definition that x has exact order β.

By choosing a subsequence if necessary, we can assume that the points t_o, $t_o+\delta_m$ both lie in the same interval $[t_k, t_{k+1}]$ of length h, and further that δ_m satisfies the condition $\delta_m < ((K_2-\varepsilon)/K_1)^{1/(1-\beta)}.h$ (3.3.8)

Thus
$$\rho_n x(t_o+\delta_m) - \rho_n x(t_o) = \frac{1}{h}\{(x(t_o+\delta_m)-x(t_k))(t_{k+1}-\overline{t_o+\delta_m}) +$$
$$+ (\overline{t_o+\delta_m}-t_k)(x(t_o+\delta_m)-x(t_{k+1}))$$
$$- (x(t_o)-x(t_k))(t_{k+1}-t_o) - (t_o-t_k)(x(t_o)-x(t_{k+1}))\}$$
$$= \frac{1}{h}\{(x(t_o+\delta_m)-x(t_o))(t_{k+1}-t_o) - (t_o-t_k)(x(t_o+\delta_m)-x(t_o))$$
$$- \delta_m(x(t_o+\delta_m)-x(t_k)) + \delta_m(x(t_o+\delta_m)-x(t_{k+1}))\}$$

Hence $|\rho_n x(t_o+\delta_m) - \rho_n x(t_o)| \geq \frac{1}{h}\Big|\,|x(t_o+\delta_m)-x(t_o)|.h - \delta_m|x(t_{k+1})-x(t_k)|\,\Big|$

The first term $> K_2\delta_m^\beta$, and the second term is $< \frac{\delta_m}{h}.K_1 h^\beta = K_1(\frac{\delta_m}{h})^{1-\beta}.\delta_m^\beta$. The inequality (3.3.8) implies that the second term is less than the first and hence

$$|\rho_n x(t_o+\delta_m) - \rho_n x(t_o)| > \varepsilon\delta_m^\beta \qquad (3.3.9)$$

Since $\delta_m \to 0$ as $m\to\infty$, we may always choose δ_m sufficiently small to satisfy (3.3.8) no matter how large n. Thus there exists $\epsilon>0$ such that for every N there exists $n>N$ and δ_m such that (3.3.9) holds. Hence $M_2[\rho_n x;\beta] \not\to 0$ as $n\to\infty$, and we have the required result:

$$\|x-P_n x\|_\beta \not\to 0 \text{ as } n\to\infty \text{ for } x(t) \text{ of exact order } \beta .$$

In the following section we wish to apply these results on a contour L. From [33] we assume L a simple, smooth contour. This implies that the curve has a continuously differentiable parameterization. The direction corresponding to an increase of the parameter, s, is considered the positive direction of the curve. The smooth contour is rectifiable [33, p. 10]. For any two points t_1,t_2 on L we have $0<k_o\le|t_2-t_1|/|s_2-s_1|\le 1$, where k_o is a constant and s_i is the arc coordinate corresponding to the point t_i, i=1,2 [33, p. 47]. For the (closed) contour we use $t_0=t_{mn}$. If the interpolation nodes are determined on the arc coordinate, then it is clear from the above inequality that all the results of this section hold, with the possible exception of a constant factor, on the contour L.

3.4 Operator Convergence in Hölder Spaces

With these results on the convergence of polynomial approximations in a Hölder space, the convergence of approximations to integral operators on a Hölder space are considered, similar to the approach in section 2.3.

3.4.1 Operator with a Hölder Continuous Kernel

The integral operator $Kx(s) = \int_0^1 k(s,t)x(t)\,dt$ maps a bounded function (and in particular a function in $H(\beta)$) into the space $H(\beta)$ if

$\int_0^1 |k(s,t)|dt = g(s)\varepsilon H(\beta)$. This will hold in particular if $k_t(s)\varepsilon H(\beta)$. P_n will be piecewise polynomial operator.

Lemma 3.4.1

Let K be an integral operator on C such that $k_t(s)\varepsilon H(\beta)$. Let the approximate operator K_n be obtained by approximate product integration. Then the approximate operators converge pointwise in $H(\beta)$

Proof:

$$\|K_n x - Kx\|_\beta = \|K(P_n x - x)\|_\beta$$

$$= M_1\left[\int_0^1 k(s,t)(P_n x(t) - x(t))dt\right] + M_2\left[\int_0^1 k(s,t)(P_n x(t) - x(t))dt;\beta\right]$$

$$\leq \left\{\max_s \int_0^1 |k(s,t)|dt + \sup_{0<|s_1-s_2|\leq 1} \frac{\int_0^1 |k(s_1,t)-k(s_2,t)|dt}{|s_1-s_2|^\beta}\right\} C.\omega(x,h)$$

$= C.\|K\|_\beta w(x,h)$, where C is constant depending on the order of interpolation and $\|K\|_\beta$ exists since $k_t \varepsilon H(\beta)$. Since $x\varepsilon C$ the result follows.

It is clear that convergence is uniform on an equicontinuous set. Since $Kx\varepsilon H(\beta)$ implies that $\{Kx:\|x\|\leq 1\}$ is an equicontinuous set, and $\{K_n x:\|x\|\leq 1\}$ is collectively equicontinuous, by lemma 3.4.1 we have

$$\|(K_n - K)Kx\|_\beta \leq C.\|K\|_\beta\ \omega(Kx,h) \leq C.\|K\|_\beta \sup_{\|x\|\leq 1} \omega(Kx,h)$$

Thus $\|(K_n - K)K\|_\beta \to 0$ as $n\to\infty$ and similarly $\|(K_n - K)K_n\|_\beta \to 0$.

These results follow exactly the same for $x\varepsilon H(\beta)$, and thus these approximations satisfy Anselone's convergence theorem in the space $H(\beta)$.

Although the above approximation scheme demonstrates the desired convergence properties, it is not a very convenient process because of the

necessity of evaluating all the integrals for the coefficients of the approximate operator. A more convenient scheme would be the direct application of the quadrature formula to the whole integrand as in section 2.3.

Lemma 3.4.2

Suppose the kernel $k(s,t)$ belongs to the class $h(\beta)$. If the approximate operators, K_n, are obtained by composite, interpolatory quadrature rules, then $K_n x \to Kx$ uniformly in $H(\beta)$, for each $x \in C$, as $n \to \infty$

Proof:

$$M_1[K_n x - Kx] = \|\Phi(P_n k_s x - k_s x)\| \le C.\|\Phi\|[\|x\|\omega_t(h) + M\omega(x,h)] \quad \text{by } (2.3.1)$$

Similarly $$M_2[K_n x - Kx;\beta] \le \sup_{0<|s_1-s_2|\le 1} \{C.\|\Phi\|[\sup_t |k_{s_1} - k_{s_2}|/|s_1-s_2|^\beta.\omega(x,h) + \|x\|\omega(k_{s_1} - k_{s_2},h)/|s_1-s_1|^\beta]$$

Now $|k(s_1,t+h) - k(s_2,t+h) - k(s_1,t) + k(s_2,t)| \le \omega(k_{s_1},h) + \omega(k_{s_2},h) \le 2\sup_s \omega(k_s,h)$

But the expression is also bounded by $\omega(k_{t+h},|s_1-s_2|) + \omega(k_t,|s_1-s_2|) \le 2\sup_t \omega(k_t,|s_1-s_2|)$. Thus

$$\sup_{h\le|s_1-s_2|\le 1} \omega(k_{s_1} - k_{s_2},h)/|s_1-s_2|^\beta \le 2 \sup_s \frac{\omega(k_s,h)}{h^\beta}$$

and

$$\sup_{0<|s_1-s_2|<h} \omega(k_{s_1} - k_{s_2},h)/|s_1-s_2|^\beta \le 2 \sup_{0<|s_1-s_2|<h} \sup_t \frac{\omega(k_t,\delta)}{\delta^\beta}$$

Since $k \in h(\beta)$, it is clear that $k_s \in h(\beta)$ uniformly with respect to s and $k_t \in h(\beta)$ uniformly with respect to t. Thus

$$\sup_{0<|s_1-s_2|\leq 1} \omega(k_{s_1}-k_{s_2},h)/|s_1-s_2|^\beta \leq 2\max(\sup_s \frac{\omega(k_s,h)}{h^\beta}, \sup_t \frac{\omega(k_t,h)}{h^\beta}) \leq \varepsilon(h) \to 0 \text{ as } h\to 0 \quad (3.4.1)$$

Then $M_2[K_nx-Kx;\beta] \leq C\|\Phi\|\{M_2[k;\beta]\,\omega(x,h)+\|x\|.\varepsilon(h)\}$ and

$$\|K_nx-Kx\|_\beta = M_1[K_nx-Kx]+M_2[K_nx-Kx;\beta]$$

$$\leq C.\|\Phi\|\{\|k\|_\beta\,\omega(x,h)+\|x\|.\varepsilon(h)\}\to 0 \text{ as } h=\frac{1}{n}\to 0$$

That is $\|K_nx-Kx\|_\beta \to 0$ as $n\to\infty$ for each fixed $x\in C$ if the sets $\{k_s\}$, $\{k_t\}$ are both equi-$H(\beta)$. (Compare lemma 2.3.1)

As in lemma 2.3.1, it is clear that convergence of the operators will be uniform on each bounded, equicontinuous set of operands. It is trivially verified that $\{Kx:\|x\|\leq 1\}$ is precompact and $\{K_nx:\|x\|\leq 1, n\geq 1\}$ is collectively precompact for $k\in h(\beta)$. Thus $\|(K_n-K)K\|_\beta\to 0$ and $\|(K_n-K)K_n\|_\beta\to 0$ as $n\to\infty$ as before. Hence although it can be shown that K is compact in $H(\beta)$ (Muskhelishvili [33, p. 133]), these remarks and the above lemma imply that these approximations satisfy Anselone's theory and the convergence theorem 1.3.2.

It is clear that computable error bounds are not particularly practical in this case. However the Fredholm equation in $H(\beta)$ may also be regarded as an equation in the larger space C. Further the integral operator transforms $x\in C$ into $Kx\in H(\beta)$. Hence we may apply the theory to the equation in C and obtain computable error bounds for $\|x-x_n\|$. The following lemma shows that the sequence $\{x_n\}$ does in fact converge in the space $H(\beta)$ as required but the rate of convergence in the $H(\beta)$ norm is again difficult to estimate.

Lemma 3.4.3

If the Fredholm equation, $x-Kx=y$, is in the space $H(\beta)$ and x_n is the solution in the space C of the approximate equation, then x_n converges to x in the $H(\beta)$ norm if $\{k_t\},\{k_s\}$ are both equi-$H(\beta)$.

Proof:

We have $x_n-K_nx_n=y$; $x-Kx=y$. Thus

$$x_n-x = Kx-K_nx_n = (K-K_n)x + K_n(x-x_n) \tag{3.4.2}$$

By lemma 3.4.2, $\|(K-K_n)x\|_\beta \to 0$ for $x\in C$ if $\{k_s\},\{k_t\}$ are equi-$H(\beta)$

$$\begin{aligned}\|K_n(x-x_n)\|_\beta &= M_1[K_n(x-x_n)]+M_2[K_n(x-x_n);\beta] \\ &\le \sup_s|\Phi P_nk_s(x-x_n)| + \sup_{0<|s_1-s_2|\le 1}|\Phi P_n(k_{s_1}-k_{s_2})(x-x_n)|/|s_1-s_2|^\beta \\ &\le \|\Phi\|\|L_m\|\|k\|\|x-x_n\|+\|\Phi\|\|L_m\|M_2[k;\beta]\|x-x_n\|, \text{ since } \{k_t\} \text{ is equi-}H(\beta) \\ &\le \|\Phi\|\|L_m\|\|k\|_\beta\|x-x_n\|\to 0 \text{ as } n\to\infty \text{ since } \|x-x_n\|\to 0 \text{ in the space C by}\end{aligned}$$

theorem 1.3.2.

Hence taking norms in (3.4.2)

$$\|x_n-x\|_\beta \le \|(K-K_n)x\|_\beta + \|K_n(x-x_n)\|_\beta \to 0 \text{ as } n\to\infty.$$

Computable error bounds for the second term may be obtained using theorem 1.3.2, but computable error bounds for the first term are again a problem as in lemma 3.4.2.

3.4.2 The Hilbert Operator

Various schemes have been proposed for the numerical evaluation of

principal value integrals (e.g., Longman [28], Delves [14]). If the singularity varies we have the case of principal value integrals with a Cauchy kernel, in which we are interested. The methods of Bareiss and Neuman [9] and Stewart [42] both consider the finite Hilbert transform (over a finite interval on the real line). Appropriate modifications are made for the end-points.

Since Stewart's method is essentially an application of the method of approximate product integration, the primary question is that of the convergence of a sequence of Cauchy integrals. It is easily seen that if $\{f_n\}$ is a sequence of functions in a bounded set in a Hölder space then the (pointwise) limit of the sequence also belongs to this same bounded set. This, in conjunction with the Privalov theorem (theorem 3.1.1), asserts that the limit of a bounded sequence of Hölder functions is Cauchy integrable. However, it is not true in general that the limit of a sequence of Cauchy integrals is the Cauchy integral of the (uniform) limit of the sequence, or even that the uniform limit itself is Cauchy integrable. An example is given by Stewart [42].

Definition [42]:

The sequence $\{f_n\}$ of functions on $[a,b]$ is strongly secant convergent on the subset $[[a,b]] = \{(s,t),\ a\le s,t\le b,\ s\neq t\}$ iff the corresponding sequence $\{F_n\}$ of difference quotients of $\{f_n\}$ is uniformly convergent on $[[a,b]]$ where

$$F_n(s,t) = (f_n(s)-f_n(t))/(s-t).$$

Note that the limiting case of the difference quotient is the derivative of the function and thus in a formal sense this condition requires the uniform convergence of the sequence of derivatives of f_n. However it is

immediately evident that strong secant convergence is merely convergence in the $H(1)$ norm (compare section 3.3).

Stewart's principal result states that if $\{p_n\}$ is a sequence of polygonal approximations to a continuously differentiable function, f, then the sequence $\{p_n\}$ is strongly secant convergent and hence the sequence $\{Hp_n(s)\}$ converges uniformly to $Hf(s)$ where H is the Hilbert operator. In the present notation this may be restated as the fact that if $f \in C'$ then the polygonal approximations, $\{P_n f\}$, converge to f in the $H(1)$ norm. This follows at once from section 3.3 since a simple calculation shows that

$$M_2[f-P_n f;1] = M_2[\rho_n f;1] \leq \max_t \left|\frac{d}{dt}\rho_n f(t)\right| \leq \omega(f';h),$$

and thus

$$\|f-P_n f\|_1 = \|\rho_n f\|_1 = M_1[\rho_n f]+M_2[\rho_n f;1] \leq \omega(f,h)+\omega(f';h)$$

where $\omega(f';h)$ is the modulus of continuity of the derivative of f.

Stewart concludes that such a sequence of approximations to a Cauchy integral will converge uniformly if f is "sufficiently smooth"--a result effectively obtained in Hardy's second paper [19], although not exactly in this form. However, the transformed functions have greater smoothness properties than merely being continuous and so uniform convergence does not ensure convergence in the norm of the range space of the operator, as required for the abstract convergence theorem. Thus considering an approach for the Hilbert operator, H, in a Hölder space $H(\beta)$ similar to that in section 2.5, we show first that the operator is bounded.

<u>Lemma 3.4.4</u>

Let H denote the Hilbert operator on the space $H(\beta)$, $0<\beta<1$. Then,

$\|H\|_\beta = \sup_{\|x\|_\beta \le 1} \|H_x\|_\beta \le C$, where C is a constant depending only on the contour of integration and the index β.

Proof:

Let the integration be over a closed contour, L, as in the Privalov theorem. Write, as in the proof of this theorem

$$Hx(s) = \int_L \frac{x(t)}{t-s}\,dt = \pi i\, x(s) + \int_L \frac{x(t)-x(s)}{t-s}\,dt. \quad \text{Then}$$

$$M_1[Hx] = \sup_s |Hx(s)| \le \pi\, M_1[x] + C_1 M_2[x;\beta] \quad \text{where } C_1 = \sup_s \int_L |t-s|^{\beta-1}|dt|$$

Similarly from the proof of the Privalov theorem we obtain

$M_2[Hx;\beta] \le C_2 M_2[x;\beta]$, where C_2 is a positive constant independent of x but dependent on the shape of the contour L. Thus

$$\|Hx\|_\beta = M_1[Hx] + M_2[Hx;\beta] \le C(M_1[x] + M_2[x;\beta]) = C\|x\|_\beta \quad \text{where } C = \max(\pi, C_1 + C_2).$$

Theorem 3.4.5

Let H be a Hilbert operator on $H(\beta)$. Let the approximate operator, H_n, be obtained by approximate product integration (polynomial interpolation of the operand). Then $H_n \to H$ pointwise in $H(\beta)$ for each $x \in h(\beta)$, and uniformly on each equi-$H(\beta)$ set of operands.

Proof:

$$\|Hx - H_n x\|_\beta = \|H(x - P_n^m x)\|_\beta \le \|H\|_\beta \|x - P_n^m x\|_\beta \le K(m)\|H\|_\beta\, \omega(x,h)/h^\beta$$

using lemma 3.3.3. Since $\|H\|_\beta$ is bounded by the previous lemma, the approximate quadratures converge in this norm for each $x \in h(\beta)$. Trivially the convergence is uniform over an equi-$H(\beta)$ set.

In particular we have convergence in this norm if $x\in H(\alpha) \subset H(\beta)$, $\alpha>\beta$ since then $\omega(x,h)/h^{\beta} \leq Bh^{\alpha-\beta}$, B constant. Alternatively, for $x\in H(\beta)$, we may regard this as indicating convergence in any coarser topology.

Corollary 3.4.6

$H_n \to H$ pointwise in $H(\gamma)$ for each $x\in h(\beta)$, $\gamma<\beta$.

Proof:

$$\|Hx-H_nx\|_{\gamma} \leq M_1[Hx-H_nx]+M_2[Hx-H_nx;\gamma] \leq M_1[Hx-H_nx]+M_2[Hx-H_nx;\beta] = \|Hx-H_nx\|_{\beta} \text{ for } \gamma>\beta$$

Stewart's result is a particular case of this corollary where $H(\gamma) = h(0) = C$.

The reciprocal relation of the Hilbert transform implies $H^{-1} = -H$. Thus $\|H^{-1}\|_{\beta} = \|H\|_{\beta} < C$ by lemma 3.4.4.

Theorem 3.4.7

The approximate operators, H_n, do not converge pointwise in the $H(\beta)$ norm if $x\in H(\beta)$ has exact order β.

Proof:

$\|P_nx-x\|_{\beta} = \|H^{-1}(H(P_nx-x))\|_{\beta} \leq \|H^{-1}\|_{\beta}\|H_nx-Hx\|_{\beta}$. From lemma 3.4.4 it is clear that $\|H\|_{\beta} \neq 0$, since $\|Hx\|_{\beta} = \pi$ for $x(t) \equiv 1$. Thus $\|H_nx-Hx\|_{\beta} \geq \frac{1}{C}\cdot\|P_nx-x\|_{\beta} \nrightarrow 0$ as $n\to\infty$, for $x\in H(\beta)$ of exact order β by lemma 3.3.5.

To conclude this section we consider the most direct method for the numerical evaluation of the Hilbert transform. Bareiss and Neumann [9] propose the "method of symmetric pairing" in which the principal value integral is reduced to an unconditionally convergent integral by "pairing" the function values of the integrand for the arguments that are symmetric with respect to the singularity. Thus

$$Hx(s) = P\int_a^b \frac{x(t)}{t-s}dt = \lim_{\varepsilon\to 0}\left\{\left(\int_a^{s-\varepsilon} + \int_{s+\varepsilon}^b\right)\frac{x(t)}{t-s}dt\right\}$$

$$= \lim_{\varepsilon\to 0}\left\{\int_\varepsilon^m \frac{x(s+u)-x(s-u)}{u}du + \sigma\int_m^M \frac{x(s+\sigma u)}{u}du\right\}$$

where $m=\min(s-a,b-s)$, $M=\max(s-a,b-s)$, $\sigma = \begin{cases} +1 & \text{if } s-a<b-s \\ -1 & \text{if } s-a>b-s \end{cases}$

The trapezoidal and mid-point rules are applied to the resulting unconditionally convergent integrals (effectively assuming the integrand constant in the first interval for the improper integral). Thus expressions are obtained for $Hx(a+kh)$ and $Hx(a+(k+\frac{1}{2})h)$ respectively, where $h = (b-a)/n$. Thus this is essentially an application of the process of ignoring the singularity considered in section 2.5. That is, for the trapezoidal rule, the function $Hx(s)$ is evaluated at the rational points $a+kh$ by a compound trapezoidal rule with the singularity at zero ignored. (For the purposes of this formal discussion we may assume (as is done in [9]) that x vanishes at the end-points in order to avoid the end-point problem).

However in [9] the authors do not consider whether these approximations converge to the integral for each k, nor whether the interpolation of these points converges to $Hx(s)$ as $n\to\infty$. By applying Miller's results [32] we show that the convergence is uniform in s.

<u>Lemma 3.4.8</u>

Let the approximate operator, H_n, be defined from the interpolation of the points obtained by applying the method of symmetrical pairing with a trapezoidal rule to the Hilbert operator H. Then $\|H_nx-Hx\|\to 0$ as $n\to\infty$ for $x\in H(\beta)$, $0<\beta\leq 1$.

Proof:

Let $\bar{H}_n$ denote the operator obtained by applying the method of symmetric pairing at the rational points t_k, $k=0,1,\ldots,n$. By Miller's result [32, theorem 2], the compound interpolatory quadrature rule converges to the singular integral at the rational points, t_k, if the integrand can be dominated by a monotone integrable function in the (one-sided) neighborhood of the singularity. Since the integrand, $(x(s+u)-x(s-u))/u$, is bounded by $M_2[x;\beta].u^{\beta-1}$ for $x\in H(\beta)$, and this bound is integrable, it follows that this quadrature converges for each fixed t_k. That is $\bar{H}_n x(t_k)\to Hx(t_k)$ for each k, as $n\to\infty$, since the convergence of the remaining integral follows as in section 2.2.

We need to show that the convergence of the improper integral portion is uniform with respect to k. Using the notation of Miller's result for the error [32, theorem 3], given $\epsilon>0$, we can find H such that

$$\left|\int_0^H P_n f(t)-f(t)dt\right| \le (1+K)\int^H F(t)dt < \epsilon$$

where K is a positive constant depending on the integration rule and F(t) is the majorizing monotone integrable function for the integrand f. Further, we can then choose h sufficiently small so that for an arbitrary point $t_k = kh$, using theorem 2.2.4,

$$\left|\int_H^{kh} P_n f(t)-f(t)dt\right| \le \omega(f,\tfrac{1}{n})(kh-H) \le \omega(f,\tfrac{1}{n})((b-a)/2-H) < \epsilon$$

where the modulus of continuity of f is calculated over the interval $[H,(b-a)/2]$, where the integrand is continuous. That is, for n sufficiently large, $\left|\int_0^{kh} P_n f(t)-f(t)dt\right| < 2\epsilon$, independent of k.

For the integral above, $F(t) = M_2[x;\beta]\,t^{\beta-1}$ and $\omega(f,\frac{1}{n}) =$ $\omega((x(s+u)-x(s-u))/u,1/n) \le (M_2[x;\beta]h^{\beta} + M_1[x].h/H).2/H$. Thus, in this case, $\max_{k\le n}|\bar{H}_n x(t_k)-Hx(t_k)| < 2\varepsilon$ for n sufficiently large.

The corresponding approximate functions are obtained by interpolating the points $\bar{H}_n x(t_k)$, $k=0,1,\ldots,n$. This gives the piecewise linear function $\varphi_o^{-1}\bar{H}_n x = H_n x(s)$, where φ_o^{-1} is the linear operator from the space, m_n, of bounded sequences of length n, to the space, $\tilde{C}$, of piecewise linear functions. Hence in the usual sup norm we obtain

$$\begin{aligned} \|H_n x-Hx\| &\le \|\varphi_o^{-1}\bar{H}_n x-P_n Hx\| + \|P_n Hx-Hx\| \qquad (3.4.3) \\ &\le \max_{k\le n}|\bar{H}_n x(t_k)-Hx(t_k)| + (Hx,\tfrac{1}{n}), \text{ since } Hx\in C. \end{aligned}$$

Hence $\|H_n x-Hx\|\to 0$ as $n\to\infty$ and we have pointwise convergence of these operators in the sup norm for each $x\in H(\beta)$.

However uniform convergence of approximations to the transformed function was obtained previously (corollary 3.4.6), and this does not imply convergence in the $H(\beta)$ norm. It was necessary to assume $x\in H(\beta)$ to obtain this result and the error bounds above involve the $H(\beta)$ norm of x. Compare theorem 3.4.7.

Lemma 3.4.9

The approximations obtained by the method of symmetric pairing do not converge pointwise in the $H(\beta)$ norm if $x\in H(\beta)$ has exact order $\beta<1$.

Proof:

Considering the inequality (3.4.3) in the $H(\beta)$ norm, we have in an alternate form

$$\|P_n Hx - Hx\|_\beta \le \max_{k\le n} |\bar{H}_n x(t_k) - Hx(t_k)| + \|H_n x - Hx\|_\beta$$

By lemma 3.2.1, if x has exact order β then Hx is also of exact order β. Thus by lemma 3.3.5 $\|P_n Hx - Hx\|_\beta \nrightarrow 0$ as $n\to\infty$. Hence the above inequality implies $\|H_n x - Hx\|_\beta \nrightarrow 0$.

That is a direct approximation method leads to the same type of difficulties as found for approximate product integration in theorem 3.4.7 and, in general, $\|H_n x - Hx\|_\beta \nrightarrow 0$ as $n\to\infty$ for $x\in H(\beta)$.

3.4.3 The General Operator with Cauchy Kernel

Consider the case of the general singular operator, K, with Cauchy kernel, $k(s,t)/(t-s)$. We consider two approximation schemes. Firstly where the kernel is integrated exactly and approximate product integration is applied in a form similar to lemma 3.4.1. Secondly where only the "singular part" of the kernel is integrated exactly and the continuous part of the integrand is interpolated for the approximate product integration scheme.

The kernel is not absolutely integrable and it is first necessary to establish that K is a bounded operator on the Hölder space. This depends on the generalized Privalov theorem where the density function depends on a parameter [33, p. 49], [38, p. 443]. Pogorzelski's result shows that if $\varphi(u,t)\in H(\alpha,\beta)$, $0<\alpha\le 1$, $0<\beta<1$, then, on the smooth contour L, the singular generalized Cauchy integral $\Phi(u,s) = \int_L \frac{\varphi(u,t)}{t-s}dt$ satisfies $\Phi(u,s)\in H(\alpha',\beta)$ where α' is an arbitrary positive constant less than α. Thus in the case that $u=s$ we have $\Phi(s) = \Phi(s,s)\in H(\alpha')$ for $\alpha\le\beta$ and $\Phi(s)\in H(\beta)$ for $\alpha>\beta$. Hence in order that the operator K maps the space $H(\beta)$ into the space $H(\beta)$ it is necessary to require that $k(s,t)\in H(\alpha,\gamma)$, where $\alpha>\beta$ and $\gamma\ge\beta$. Note that this condition

implies that $k_s \varepsilon H(\gamma)$ uniformly with respect to s, and $k_t \varepsilon H(\alpha)$ uniformly with respect to t.

Lemma 3.4.10

Under the assumed conditions, the operator K is a bounded, linear operator on $H(\beta)$. That is $\|K\|_\beta = \sup_{\|x\|_\beta \le 1} \|Kx\|_\beta \le C$ for some constant C.

Proof:

Following the proof of lemma 3.4.4, but using the generalized Privalov theorem in this case

$$M_1[Kx] \le \pi\, M_1[kx] + C_1\, M_2[(kx)_s;\beta]\text{, and}$$

$$M_2[Kx;\beta] \le \pi M_2[(kx)_t;\beta] + C_2 M_2[(kx)_s;\beta] + C_3\{M_2[(kx)_s;\beta] + M_2[(kx)_t;\alpha]\}$$

where C_3 is also a constant independent of k,x but depending on the contour L. Thus with the bounds

$M_1[kx] \le M_1[k].M_1[x]$;

$M_2[(kx)_t;\beta] \le C_4\, M_2[(kx)_t;\alpha] \le C_4\, M_2[k_t;\alpha].M_1[x]$ since $\alpha > \beta$;

$M_2[(kx)_s;\beta] \le M_1[k].M_2[x;\beta] + M_1[x].M_2[k_s;\beta]$, we obtain

$$\begin{aligned}\|Kx\|_\beta &= M_1[Kx] + M_2[Kx;\beta] \\ &\le \{\pi\, M_1[k] + (C_3+\pi C_4)M_2[k_t;\alpha] + (C_1+C_2+C_3)M_2[k_s;\beta]\}M_1[x] \\ &\qquad + (C_1+C_2+C_3)M_1[k].M_2[x;\beta] \\ &\le C_5\|k\|_{\alpha,\beta}\, M_1[x] + C_6\|k\| M_2[x;\beta]\end{aligned}$$

where $\|k\|$ denotes the upper bound of the function $k(s,t)$,

$\|k\|_{\alpha,\beta}$ denotes $\|k\|+M_2[k_t;\alpha]+M_2[k_s;\beta]$, since $k_s \epsilon H(\gamma)$ implies $k_s \epsilon H(\beta)$, $\gamma \geq \beta$,

$C_5 = \max(\pi, C_3+\pi C_4, C_1+C_2+C_3)$ and $C_6 = C_1+C_2+C_3$.

Thus $\|Kx\|_\beta \leq C\|x\|_\beta$ where $C = \max(C_5\|k\|_{\alpha,\beta}, C_6\|k\|)$.

Lemma 3.4.11

Let K be a bounded integral operator on $H(\beta)$ with a Cauchy kernel, $k(s,t)/(t-s)$ where $k \epsilon H(\alpha,\beta)$, $\alpha > \beta$. Let the approximate operator, K_n, be obtained by approximate product integration where the operand is approximated. Then the sequence of approximate operators converges pointwise on $H(\beta)$ for each $x \epsilon h(\beta) \subset H(\beta)$, and uniformly on each equi-$H(\beta)$ set of operands.

Proof:

$$\|K_n x - Kx\|_\beta = \|K(P_n x - x)\|_\beta \leq \|K\|_\beta \|P_n x - x\|_\beta \leq \|K\|_\beta \cdot K(m) \sup_{0<\delta\leq h} \frac{\omega(x,\delta)}{\delta^\beta}$$ by

lemma 3.3.3. Since $\|K\|_\beta$ is bounded by the previous lemma, it is clear that the approximate operators converge pointwise in $H(\beta)$ for each $x \epsilon h(\beta)$, and in fact converge uniformly on each equi-$H(\beta)$ set of operands.

From this lemma it cannot be asserted that the approximate operators converge pointwise for $x \epsilon H(\beta)$. But we can make the statement:

Lemma 3.4.12

Suppose that $k(s,s)$ does not vanish anywhere on the contour L. Then the approximate operators, K_n, of the previous lemma do not converge pointwise in the $H(\beta)$ norm if $x \epsilon H(\beta)$ has exact order β.

Proof:

Write

$$k(s,s)\int_L \frac{P_n x(t)-x(t)}{t-s}dt = \int_L \frac{k(s,t)}{t-s}(P_n x(t)-x(t))dt - \int_L \frac{k(s,t)-k(s,s)}{t-s}(P_n x(t)-x(t))dt$$

Thus writing $b(s) = k(s,s)$ we have

$$\|b.H(P_n x-x)\|_\beta \le \|K_n x-Kx\|_\beta + \|L(x-P_n x)\|_\beta \tag{3.4.4}$$

where H is the Hilbert operator and L is the integral operator with the weakly singular kernel $\ell(s,t) = (k(s,t)-k(s,s))/(t-s)$.

The term with the operator L converges by the argument of lemma 3.4.1. Although the kernel does not belong to $H(\beta)$, the same argument gives

$$\|L(x-P_n x)\|_\beta \le \{\max_s \int_L |\ell(s,t)||dt| + \sup_{0<|s_1-s_2|} \int_L \frac{|\ell(s_1,t)-\ell(s_2,t)|}{|s_1-s_2|^\beta}|dt|\}\omega(x,h).$$

Since $\ell(s,t)$ is weakly singular the first term in the braces is bounded. By Pogorzelski's lemma [38, lemma 1, p. 492] the kernel, $\ell(s,t)$ may be expressed in the form $(k(s,t)-k(s,s))/(t-s) = k^*(s,t)/|t-s|^{1-\epsilon}$, where ϵ is an arbitrary positive number less than β and $k^*(s,t)\epsilon H(\alpha-\epsilon,\beta-\epsilon)$. Thus the second term in the braces is bounded by

$$\sup_{0<|s_1-s_2|} \frac{1}{|s_1-s_2|^\beta}\left\{\int_L |k^*(s_1,t)|.\left|\frac{1}{|t-s_1|^{1-\epsilon}}-\frac{1}{|t-s_2|^{1-\epsilon}}\right||dt| + \right.$$

$$\left. + \int_L |k^*(s_1,t)-k^*(s_2,t)|\frac{|dt|}{|t-s_2|^{1-\epsilon}}\right\}$$

$$\le \sup_{0<|s_1-s_2|}\left\{|s_1-s_2|^{1-\beta-\epsilon}.D.\int_L \frac{|dt|}{|t-s_1|^{1-\epsilon}|t-s_2|^{1-\epsilon}} + \right.$$

$$\left. + |s_1-s_2|^{\alpha-\epsilon-\beta}.M_2[k_t^*;\alpha-\epsilon]\int_L \frac{|dt|}{|t-s_2|^{1-\epsilon}}\right\}$$

where D is the absolute bound for k^*. These integrals are clearly bounded (compare Muskhelishvili [33, p. 136] for a discussion of the first integral), and thus by choosing $\epsilon \leq \min(1-\beta, \alpha-\beta)$, the entire term is bounded. It is thus concluded that $\|L(x-P_n x)\|_\beta \leq E\,\omega(x,h)$ for a constant E, and hence this term converges to zero as $n\to\infty$ for $x\in H(\beta)$.

On the other hand, by theorem 3.4.7, $\|H_n x - Hx\|_\beta \not\to 0$ for x of exact order β. Since we know (corollary 3.4.6) that $\|H_n x - Hx\| \to 0$, this implies $M_2[H_n x - Hx;\beta] \not\to 0$ as $n\to\infty$. Thus there exists $\epsilon>0$ such that for all N, there exists $n>N$ and two points s_1, s_2 such that $|\sigma_n(s_1)-\sigma_n(s_2)|/|s_1-s_2|^\beta > \epsilon$, where $\sigma_n(s) = H_n x(s) - Hx(s)$. Using this result we obtain

$$\frac{|b(s_1)\sigma_n(s_1)-b(s_2)\sigma_n(s_2)|}{|s_1-s_2|^\beta} = \left| b(s_1)\frac{\sigma_n(s_1)-\sigma_n(s_2)}{|s_1-s_2|^\beta} + \frac{b(s_1)-b(s_2)}{|s_1-s_2|^\beta}\cdot\sigma_n(s_2)\right| \geq$$

$$\geq \left| |b(s_1)|\frac{|\sigma_n(s_1)-\sigma_n(s_2)|}{|s_1-s_2|^\beta} - \frac{|b(s_1)-b(s_2)|}{|s_1-s_2|^\beta}|\sigma_n(s_2)|\right|$$

The right side is greater than $(|b(s_1)|\epsilon - M_2[k;\beta]\eta)$ where $|\sigma_n(s_2)|<\eta$, an arbitrarily small constant, for n sufficiently large since $\|\sigma_n\|\to 0$ as $n\to\infty$. Since $b(s_1) \neq 0$, we deduce that, for sufficiently large N, $M_2[b(H_n x-Hx);\beta] \geq |b(s_1)\sigma_n(s_1)-b(s_2)\sigma_n(s_2)|/|s_1-s_2|^\beta > \epsilon/2$, and hence $\|b.(H_n x-Hx)\|_\beta \not\to 0$ for $x\in H(\beta)$ of exact order β.

In view of these two results and inequality (3.4.4) it is concluded that $\|K_n x - Kx\|_\beta \not\to 0$ for $x\in H(\beta)$ of exact order β.

The procedure in lemma 3.4.11 involves evaluations of the integral $\int_L k(s,t)/(t-s)dt$ in order to obtain the coefficients in the approximate form. A simpler approximate form could be obtained by interpolating the whole numerator of the general singular operator and integrating only the "singular part".

Lemma 3.4.13

Let K be a singular integral operator on $H(\beta)$ with a Cauchy kernel $k(s,t)/(t-s)$. Suppose $k(s,t) \in H(\alpha,\alpha) = H(\alpha)$, $\alpha > \beta$, and let the approximate operators, K_n, be obtained by approximate product integration where the singular part, $1/(t-s)$, is integrated exactly. Then the sequence of approximate operators converges pointwise in the $H(\beta)$ norm for each $x \in h(\beta)$, and uniformly on each bounded equi-$H(\beta)$ set of operands.

Proof:

From the remarks at the beginning of this section it follows that K is a bounded, linear operator from $H(\beta)$ to $H(\beta)$. Using the generalized Privalov theorem as in the proof of lemma 3.4.10,

$$M_1[K_n x - Kx] \le \pi\, M_1[P_n k_s x - k_s x] + C_1\, M_2[(P_n k_s x - k_s x)_s; \beta], \text{ and}$$

$$M_2[(K_n x - Kx);\beta] \le (\pi C_4 + C_3) M_2[(P_n k_s x - k_s x)_t; \alpha'] + (C_2 + C_3) M_2[(P_n k_s x - k_s x)_s; \beta]$$

where we may choose an arbitrary α' such that $\beta < \alpha' < \alpha$. We have the following bounds.

$$M_1[P_n k_s x - k_s x] \le C\{\|x\| \sup_s \omega(k_s, h) + \|k\|\omega(x,h)\} \quad \text{by (2.3.1)}$$

$$M_2[(P_n k_s x - k_s x)_s;\beta] \le K_1(m) \sup_{0<\delta\le h} \frac{\omega(k_s x, \delta)}{\delta^\beta} \qquad \text{by (3.3.7)}$$

$$\le K_1(m)\{\|x\| \sup_{0<\delta\le h} \frac{\omega(k_s, \delta)}{\delta^\beta} + \|k\| \sup_{0<\delta\le h} \frac{\omega(x,\delta)}{\delta^\beta}\} \quad \text{by (2.3.1)}$$

$$M_2[(P_n k_s x - k_s x)_t;\alpha'] \le C \sup_{0<|s_1-s_2|} \frac{\omega((k_{s_1}-k_{s_2})x,h)}{|s_1-s_2|^{\alpha'}}$$

$$\le C \sup_{0<|s_1-s_2|} \{\|x\| \frac{\omega(k_{s_1}-k_{s_2},h)}{|s_1-s_2|^{\alpha'}} + \sup_t \frac{|k_{s_1}-k_{s_2}|}{|s_1-s_2|^{\alpha}} \omega(x,h)\} \quad \text{by (2.3.1)}$$

$$\le C'\{2\|x\| . M_2[k;\alpha] h^{\alpha-\alpha'} + M_2[k;\alpha]\omega(x,h)\} \quad \text{by (3.4.1)}$$

Thus, collecting all expressions

$$\|K_n x - Kx\|_\beta \le M_1[K_n x - Kx] + M_2[K_n x - Kx;\beta]$$

$$\le C_5\|x\| M_2[k;\alpha] h^{\alpha-\alpha'} + C_6\|k\|_\alpha \sup_{0<\delta\le h} \frac{\omega(x,\delta)}{\delta^\beta}$$

where C_5, C_6 are constants depending only on the order, m, of the interpolating polynomial and the contour L.

Hence the sequence of approximate operators converges pointwise in $H(\beta)$ for each $x \in h(\beta)$, and uniformly over each bounded, equi-$H(\beta)$ set of operands.

Thus again we cannot assert pointwise convergence for $x \in H(\beta)$. However, as in corollary 3.4.6, we trivially have convergence in any coarser topology. In particular

Corollary 3.4.14

The sequence $\{K_n x(s) - Kx(s)\}$ converges in the sup norm of space C, uniformly over a bounded set in $H(\beta)$.

Proof:

Considering the bound for the M_1 term in the above lemma we are at liberty to choose any index $\le \beta$ in the second term.

$$
\begin{aligned}
M_1[K_n x - Kx] &\le \pi\, M_1[P_n k_s x - k_s x] + C_1' \, M_2[(P_n k_s x - k_s x)_s ; \beta/2] \\
&\le C_2 \cdot K(m) \cdot M_2[k_s x;\beta]\, h^{\beta/2} \text{ by corollary 3.3.4} \\
&\le C_2 \; K(m)\{\|x\| M_2[k_s;\beta] + \|k\| M_2[x;\beta]\}\, h^{\beta/2} \\
&\le C_3 \cdot \|k\|_\alpha \cdot \|x\|_\beta h^{\beta/2} \to 0 \text{ as } n\to\infty, \text{ uniformly for } \|x\|_\beta \le B.
\end{aligned}
$$

Although singular integral equations are usually considered in the space $H(\beta)$, the above remarks concerning pointwise convergence imply that requiring the functions to belong to $h(\beta)$ might be a more desirable hypothesis. However, it is not clear that the Privalov theorem could be extended to the necessary result that the Hilbert transform of a function in $h(\beta)$ also belongs to $h(\beta)$ (Compare the remark in section 3.2 that Hx does not necessarily belong to C for $x \epsilon C = h(0)$).

Following the analysis in Chapter 2, the remaining condition to be considered is the compactness of the operator K. By lemma 3.3.1 a subset of $h(\beta)$ is compact iff it is bounded and equi-$H(\beta)$. Further a compact set in $h(\beta)$ is clearly compact in the larger space $H(\beta)$. It follows from the above lemmas that $\|(K_n - K)K\|_\beta \to 0$ if K is a compact operator (in $H(\beta)$). Since the operator K is bounded in $H(\beta)$, it is specifically required that $\{Kx : \|x\|_\beta \le 1\}$ is an equi-$H(\beta)$ set. Since the general operator K may be expressed as the sum of a Hilbert operator and a weakly singular Fredholm operator, it suffices to consider whether the Hilbert operator satisfies this condition. But by lemma 3.2.1, if x has exact order β then Hx has exact order β. Thus, from the definition in section 2.3, $\omega(Hx,h)/h^\beta \not\to 0$ as $h\to 0$. Hence the set $\{Hx : \|x\|_\beta \le 1\}$ (and by inference $\{Kx : \|x\|_\beta \le 1\}$) is not equi-$H(\beta)$.

Hence we cannot assert either $K_n \to K$ in $H(\beta)$, or $\|(K_n - K)K\|_\beta \to 0$ and thus the hypotheses for the abstract convergence theorem 1.3.2 are not satisfied. On the other hand we do have $K_n \to K$ in the space $h(\beta)$ (by lemmas 3.4.11 and 3.4.14), but it is not clear that we can assert $Hx \in h(\beta)$ even if $x \in h(\beta)$.

The situation is similar to that of the Fredholm equation in the space R (where $C \subset R \subset M$, the space of bounded functions) discussed in section 2.4. The concept of a regular set was introduced in order to extend the region of uniform convergence. But this concept depended upon the fact that the integral operator is a positive linear functional and since the Hilbert operator is not a positive operator, it does not seem that this idea would provide a fruitful extension. This difference in behaviour from the ordinary integral operator is apparently a consequence of the property of the Cauchy principal value integral being only conditionally and not absolutely convergent.

3.5 Remarks on Investigations of a Direct Method

Gabdulhaev [17] presents some investigations of the conditions for solvability and convergence of the method of constructing solutions to singular integral equations by means of polygonal approximation. Although this justification of the method is generally based on the general approximation theory of Kantovovich [25], the form of the approximation is much the same as those considered in the preceding sections. The convergence theorem will be reviewed in the light of the difficulties that were revealed in the previous section.

Corresponding to Kantovovich's notation the polygon $P_n x(t)$ is denoted by $\tilde{x}(t)$ and the space $\tilde{H}(\beta)$ is the set of polygons corresponding to $x \in H(\beta)$.

Gabdulhaev considers the singular integral equation with a Hilbert kernel of the normal [38] type:

$$Kx(s) = a(s)x(s) + Ux(s) = a(s)x(s) + \frac{1}{2\pi}\int_0^{2\pi} b(s,t)\cot(\frac{t-s}{2})x(t)dt = y(s) \tag{3.5.1}$$

where $a(s)$, $b(s,t)$ $y(s)\epsilon H(\alpha)$, $0<\alpha\leq 1$. Note that the singular part of the kernel in the above operator has the form $\cot((t-s)/2)$, which is called the Hilbert kernel. However it is easily shown (compare Pogorzelski [38], Muskhelishvili [33]) that this operator is equivalent to an operator with a Cauchy kernel, $1/(t-s)$, over a closed contour L, which we have been investigating.

An approximate solution to equation (3.5.1) is sought in the form of a polygon of the type $\tilde{x}(s) = \sum_{k=0}^{n-1} c_k\varphi_k(s)$, where $\varphi_k(s) = (s-s_{k-1})/(s_k-s_{k-1})$, $s_{k-1}\leq s\leq s_k$; $(s_{k+1}-s)/(s_{k+1}-s_k)$, $s_k\leq s\leq s_{k+1}$; 0 elsewhere. The coefficients are determined from the system of linear algebraic equations

$$K\tilde{x}(s_i) = y(s_i) \qquad (i=0,1,\ldots,n-1)$$

The latter system can be rewritten in the form

$$a_i c_i + \sum_{k=0}^{n-1} b_{ik}c_k = y_i \quad , \quad i=0,1,\ldots,n-1 \tag{3.5.2}$$

where $a_i = a(s_i)$, $y_i = y(s_i)$, $b_{ik} = \frac{1}{2\pi}\int_0^{2\pi} b(s_i,t)\cot\frac{(t-s_i)}{2}\varphi_k(t)dt$.

We note that this approximation scheme is effectively just an application of the approximate product integration method considered in lemma 3.4.11.

3.5.1 Solvability and Convergence of the Method

Gabdulhaev proposes that, since K is a bounded, linear operator from $H(\beta)$ into $H(\beta)$, $(0<\beta<\alpha)$ (compare also lemma 3.4.10), the equation (3.5.1) can be considered as a linear functional equation of the form $Kx = ax+Ux = y$ in the space $H(\beta)$. It is supposed that under the given conditions, equation (3.5.1) has a unique solution $x^*(s)$ from some Hölder class. Then it is stated that it is known from [33] that such a solution satisfies the condition $x^*(s)\epsilon H(\delta)$ $(0<\delta<\alpha)$. The numbers α,β and δ are fixed so that $0<\beta<\delta<\alpha$. The system (3.5.2) is rewritten in the space of polygons $\tilde{H}(\beta)$ in the form of an equivalent linear functional equation

$$\tilde{K}\tilde{x} = P_n(a\tilde{x}) + P_n(U\tilde{x}) = \tilde{y} \tag{3.5.3}$$

where

$$\tilde{K}:\tilde{H}(\beta)\to\tilde{H}(\beta) \text{ and } \|\tilde{K}\|_\beta = \sup_{\|\tilde{x}\|_\beta\leq 1,\tilde{x}\epsilon\tilde{H}(\beta)} \|P_nK\tilde{x}\|_\beta \leq \|P_n\|_\beta\|K\|_\beta$$

Note that $\|P_n\|_\beta\leq 3$.

Convergence of the method is established by the theorem

Theorem 3.5.1 [17, Theorem 1, p. 219]

Suppose that an equation (3.5.1) is uniquely solvable for an arbitrary right side and that the linear operator $\tilde{K}^{-1}$ $(n\geq n_o)$ exists. Then the approximate solutions, $\tilde{x}(s)$, converge to the exact solution $x^*(s)$ in the sense of the metric $H(\beta)$. In this connection

$$\|x^*-\tilde{x}^*\|_\beta < (1+3\|K\|_\beta\|\tilde{K}^{-1}\|_\beta).3M_2(x^*;\delta).h^{\delta-\beta} \tag{3.5.4}$$

To begin the analysis of this section we note that the stated hypotheses of the equation (3.5.1) and the assumed solution space do not

seem to be consistent. If it is supposed that the solution satisfies the condition $x^*(t) \epsilon H(\delta)$, $0<\delta<\alpha$ with exact order δ, then $x^*(t)$ does not belong to any space smaller than $H(\delta)$. Thus $b(s,t)x^*(t) \epsilon H(\alpha,\delta)$ and by the generalization of Privalov's theorem [38, p. 443] (compare also section 3.4.3) $Ux^*(s) \epsilon H(\delta)$ and does not belong to any smaller space $H(\alpha)$. Further ax^* belongs to $H(\delta)$ exactly and thus $Kx^* \epsilon H(\delta)$ and not to the smaller space $H(\alpha)$. On the other hand the hypotheses give the right side, $y \epsilon H(\alpha) \subset H(\delta)$, which gives a contradiction.

However if the hypotheses on the given functions are modified to

$$a(s), y(s) \in H(\alpha) \quad ; \; b(s,t) \in H(\gamma,\alpha) \; , \; 0<\alpha<\gamma\leq 1 \tag{3.5.5}$$

then $Ux \in H(\alpha)$ for $x \epsilon H(\alpha)$ and the equation $Kx=y$ may be regarded as an equation in the space $H(\alpha)$.

Further, under these conditions, the operator K satisfies $K{:}H(\beta)\to H(\beta)$, $0<\beta\leq\alpha$ (compare section 3.4.3), and hence for $y \epsilon H(\beta)$ the equation $Kx=y$ is an equation in the space $H(\beta)$. However the condition $y \epsilon H(\gamma) \subset H(\beta)$, $0<\beta<\gamma$, leads to a contradiction as in the argument above, if it is assumed that a solution exists in $H(\beta)$ and in no smaller space.

Thus in order to determine the appropriate solution space of the equation it is necessary to determine the smallest Hölder index satisfied by the given functions. Hence if $y \epsilon H(\delta)$ then we may seek the solution $x^* \epsilon H(\delta)$ in Gabdulhaev's equation and the above convergence theorem holds. However we obtain convergence only in the coarser $H(\beta)$ topology and it is clear that theorem 3.5.1 does not assert convergence in the solution space $H(\delta)$.

The convergence in theorem 3.5.1 also depends upon the existence, and uniform boundedness for sufficiently large n, of the inverse operator

$\tilde{K}^{-1}$. Kantorovich [25, p. 546] notes that this condition will certainly hold if the operator is completely continuous (or compact), but that the analysis will still hold if this result can be established directly. It has been observed previously that we were unable to assert that the singular operator is compact in the Hölder spaces. In succeeding theorems Gabdulhaev obtains this uniform boundedness under quite stringent conditions. However the convergence theorem is still unsatisfactory under these conditions since the convergence is not in the metric of the solution space, $H(\alpha)$.

Lemma 3.5.2

The Gabdulhaev approximations do not converge in the solution space $H(\alpha)$.

Proof:

Since the functional equation (3.5.3) is equivalent to the linear system (3.5.2) we have $\tilde{x}^* = \varphi_o^{-1}\bar{x}^*$ where $\bar{x}^*$ is the solution of (3.5.2) and φ_o^{-1} is the linear operator from the space m_n to $\tilde{H}(\alpha)$. By theorem 3.5.1 we have $\|\tilde{x}^*-x^*\|_\beta \to 0$. This implies convergence in any coarser topology, and in particular uniform convergence in the space C, $\|\tilde{x}^*-x^*\|\to 0$. By further specializing this result we obtain $\max_{i\le n}|\tilde{x}^*(s_i)-x^*(s_i)|\to 0$, which may be equivalently written $\max_{i\le n}|\bar{x}^*(s_i)-x^*(s_i)|\to 0$, since $\tilde{x}^* = \varphi_o^{-1}\bar{x}^*$ is the interpolation of the points $\bar{x}^*(s_i)$, $i=0,1,\ldots,n-1$. Thus we may write (where $\varphi_i(s)$ is defined in the previous section),

$$\|P_n x^* - \tilde{x}^*\|_\alpha = \Big\| \sum_{i=0}^{n-1} (x^*(s_i) - \bar{x}^*(s_i))\varphi_i(s)\Big\|_\alpha$$

$$\le \max_{i\le n}|x^*(s_i)-\bar{x}^*(s_i)| \Big\| \sum_{i=0}^{n-1} \varphi_i(s)\Big\|_\alpha$$

$$= \max_{i\le n}|x^*(s_i)-\bar{x}^*(s_i)|, \text{ since } \sum_{i=0}^{n-1}\varphi_i(s) = P_n(1) = 1$$

The right side converges to zero by the result derived immediately above. Thus we may write

$$\|P_n x^* - x^*\|_\alpha \le \|P_n x^* - \tilde{x}^*\|_\alpha + \|\tilde{x}^* - x^*\|_\alpha \le \max_{i\le n} |x^*(s_i) - \bar{x}^*(s_i)| + \|\tilde{x}^* - x^*\|_\alpha$$

Since the first term on the right converges to zero as $n\to\infty$ and an application of lemma 3.3.5 implies that $\|P_n x^* - x^*\|_\alpha \not\to 0$ for $x \in H(\alpha)$ exactly, we conclude that $\|\tilde{x}^* - x^*\|_\alpha \not\to 0$.

The argument for non-convergence depends, as usual, on the fact that the polygonal approximations converge to the generating function only on a subset of the Hölder space. In terms of the Anselone theory it has already been noted that this method of approximation fails because of the failure of pointwise convergence of the sequence of approximate operators (lemma 3.4.12 and theorem 3.4.7) for the same reason. In the following section discussing Kantorovich's analysis, it will be seen that while pointwise convergence of the approximate operators is obtained in the approximation space $\tilde{H}(\alpha)$ (condition I), the remaining Kantorovich conditions (section 1.3.2) fail to be satisfied due to the failure of the polygonal approximations to converge in $H(\alpha)$.

From lemma 3.4.11 we cannot assert pointwise convergence of the approximate operators in $H(\alpha)$. Also the previous discussion on compactness implies that the operator U is compact in $H(\alpha)$ if it maps a bounded set in $H(\alpha)$ into $H(\gamma)$, $\gamma > \alpha$ (an equi-$H(\alpha)$ set). But the discussion of the generalized Privalov theorem at the beginning of section 3.4.3 does not imply a relation of this nature.

Thus Gabdulhaev's approach is essentially to obtain "compactness" and pointwise convergence in the coarser topology of $H(\beta)$, $\beta<\alpha$. Thus for x restricted to lie in $H(\alpha)$, we obtain $\|U_n x-Ux\|_\beta \leq \|U\|_\beta \|P_n x-x\|_\beta \to 0$ as $n\to\infty$ by lemma 3.4.11 and corollary 3.3.4. Further with x restricted to $H(\alpha)$, we have $Ux \in H(\alpha)$. Thus using lemma 3.4.10 and corollary 3.3.2, $\{Ux: \|x\|_\alpha \leq 1\}$ is equi-$H(\beta)$, and hence U is a "compact" operator in the space $H(\beta)$. Thus, formally, the abstract convergence theorem 1.3.2 applies in the space $H(\beta)$. But convergence in the $H(\beta)$ metric, as in Gabdulhaev's theorem 3.5.1, does not imply convergence in the solution space, $H(\alpha)$, of the equation. That is the $H(\beta)$ limit of the sequence of approximate solutions implies that the limit function belongs to $H(\beta)$, but it is not necessary that this function belongs to $H(\alpha)$, $\alpha>\beta$. Hence the limit of the sequence is not necessarily a solution to the equation 3.5.1.

This situation may be compared to the well-known fact that uniform convergence (in C) of a sequence of functions does not necessarily imply convergence of the sequence of derivatives (i.e., convergence in the smaller space C'). More specifically compare Loud's construction [30] where a uniformly convergent (in C) sequence of functions in H(1) converges to a function that is in $H(\beta)$ exactly. It is clear that the sequence also converges in $H(\beta)$ in this case but not in any smaller space including $H(\alpha)$ and H(1). The basic problem is that the subset $H(\alpha) \subset H(\beta)$ is not closed or complete in the $H(\beta)$ norm when the Hölder constants of the subset $H(\alpha)$ are not fixed.

3.6 General Theory of Approximate Methods

In this section Kantorovich's general theory for equations of the second kind (section 1.3.2) is considered for the case of a linear integral equation. This approach is compared to that of Anselone and the similarity

of the Kantorovich conditions to those of Anselone is demonstrated. Although Gabdulhaev's convergence theorem (theorem 3.5.1) does not directly depend on the Kantorovich convergence theorem (theorem 1.3.3), his approach is based on the Kantorovich theory. It is noted that an appeal to the Kantorovich theory does not avoid the basic problems of the Gabdulhaev approximations.

Kantorovich projects the exact equation $Nx \equiv x-Kx = y$ in the normed space X into an approximate equation $\tilde{N}\tilde{x} \equiv \tilde{x}-\tilde{K}\tilde{x} = \tilde{y}$ in the complete subspace $\tilde{X} \subset X$. Anselone's basic approach is to obtain an approximate operator, K_n, to the exact operator, K, both acting on the complete space X. By regarding the equation with the approximate operator, $N_n x \equiv x-K_n x = y$, as the approximate equation, similar results are ultimately obtained.

Consider the Fredholm equation $x-Kx = y$, with continuous kernel k, in the space $X = C$. The usual algebraic system (obtained by piecewise polynomial approximation of the integrand)

$$x(t_j) - \sum_{i=0}^{mn} A_i k(t_j,t_i)x(t_i) = y(t_j) \ , \ j=0,1,\ldots,mn \qquad (3.6.1)$$

will be regarded as an approximate functional equation in the space $\bar{X}$ of bounded sequences of length $(mn+1)$. This equation may be written in symbolic form $\bar{x}-\bar{K}\bar{x} = \varphi y$ where φ denotes the linear operation mapping X onto $\bar{X}$. We take as $\tilde{X}$ the space of piecewise polynomial functions of degree m, on the system of equi-spaced base points. Let φ_o be the linear operation providing a one-to-one mapping of the complete subspace $\tilde{X}$ onto the complete space $\bar{X}$, and φ coincides with φ_o on $\tilde{X}$. In this case condition I (section 1.3.2) becomes condition Ia [25, p. 558]

Ia: For every $\tilde{x}\epsilon\tilde{X}, \|\varphi K\tilde{x}-\bar{K}\varphi_o\tilde{x}\| \leq \bar{\eta}\|\tilde{x}\|$, where $\eta = \bar{\eta}\|\varphi_o^{-1}\|$

We now show that the conditions on the kernel which imply that the Anselone hypotheses are satisfied (theorem 2.3.4), also satisfy the Kantorovich conditions (section 1.3.2).

Condition I basically expresses the convergence of the approximate operators in the approximation space $\tilde{X}$, and in fact shows uniform convergence of the operators over a bounded set in each approximation subspace (compare lemma 2.3.1).

Lemma 3.6.1

If the approximate operators are obtained by a sequence of composite, interpolatory quadrature rules, then condition I is satisfied if $\omega_t(h)\to 0$ as $h\to 0$.

Proof:

Considering the equivalent condition Ia:

$$\|\varphi K\tilde{x}-\bar{K}\varphi_0\tilde{x}\| = \max_{j\le mn}\left|\int_0^1 k(t_j,t)\tilde{x}(t)dt - \sum_{i=0}^{mn} A_i k(t_j,t_i)\tilde{x}(t_i)\right|$$

$$\le \max_{j\le mn}\int_0^1 \left|k_{t_j}\tilde{x} - P_n^m k_{t_j}\tilde{x}\right|dt \qquad (3.6.2)$$

For an arbitrary value t in one of the n subintervals (as in lemma 2.2.2)

$$|k_s\tilde{x}(t)-P_n^m k_s\tilde{x}(t)| \le |(k_s(t)-P_n^m k_s(t))\tilde{x}(t)| + |P_n^m k_s(t).\tilde{x}(t)-P_n^m k_s\tilde{x}(t)|$$

Since $\tilde{x}(t) = P_n^m x(t)$ and thus $\tilde{x}(t_j) = x(t_j)$, and further $\sum_{j=0}^{m}\ell_j(t) = 1$, then using the notation of lemma 2.2.2

$$|P_n^m k_s(t).\tilde{x}(t)-P_n^m k_s\tilde{x}(t)| = \Big|\sum_{i=0}^{m} k_s(t_i)\ell_i(t)\sum_{j=0}^{m} x(t_j)\ell_j(t) - \sum_{i=0}^{m} k_s(t_i)x(t_i)\ell_i(t)\sum_{j=0}^{m}\ell_j(t)\Big|$$

$$= \left| \sum (k_s(t_i)-k_s(t_j))(x(t_j)-x(t_i))\ell_i(t)\ell_j(t) \right|$$

where the sum is taken over the ${}^{m+1}C_2$ terms consisting of all pairs of values i,j such that $i \neq j$. A crude bound sufficient for our purposes is given by (compare proof of lemma 3.3.3)

$$\begin{aligned} \omega(k_s,mh).2\|\tilde{x}\|\Sigma|\ell_i(t)\ell_j(t)| &\leq 2m\,\omega(k_s,h).\|\tilde{x}\|\Sigma {}^mC_i\,{}^mC_j \\ &= m\,\omega(k_s,h).\|\tilde{x}\|\Big(\sum_{i=0}^{2m} {}^{2m}C_i - \sum_{i=0}^{m} ({}^mC_i)^2\Big) \\ &= \omega(k_s,h).\|\tilde{x}\|.m(2^{2m}-(2m)!/(m!)^2) \end{aligned}$$

Hence $|k_s\tilde{x}(t)-P_n^m k_s\tilde{x}(t)| \leq \omega(k_s,h).\|\tilde{x}\|\{m\|L_m\| + m(2^{2m}-(2m)!/(m!)^2)\}$ using lemma 2.2.2. Substituting into (3.6.2)

$$\|\varphi K\tilde{x}-\bar{K}\varphi_o\tilde{x}\| \leq \|\Phi\| \max_{j\leq mn} B(m)\omega(k_{t_j},h).\|\tilde{x}\| \leq \|\Phi\|.B(m)\omega_t(h).\|\tilde{x}\|$$

where $B(m) = m\|L_m\| + m(2^{2m}-(2m)!/(m!)^2)$. Thus condition Ia is satisfied with $\bar{\eta} = \|\Phi\|.B(m).\omega_t(h)$.

The condition II is related to the compactness of the operator K. (Compare lemma 2.3.2)

Lemma 3.6.2

Condition II is satisfied with $\eta_1 = \omega_s(h)(m\|L_m\|.\|\Phi\|)$.

Proof:

We may take the element $\tilde{x}$ to be $P_n^m Kx$. Thus

$$\|Kx-\tilde{x}\| = \|Kx-P_n^m Kx\| \leq m\|L_m\|\omega(Kx,h) \text{ by lemma 2.2.2}$$

$$\leq m\|L_m\| \, \|\Phi\|\omega_s(h)\|x\| \text{ by lemma 2.3.2}$$

Finally by applying a similar argument to the function y_i we find $P_n^m y = \tilde{y}\epsilon\tilde{X}$ such that $\|y-\tilde{y}\| \leq m\|L_m\|\omega(y,h)$, and condition III is satisfied with $\eta_2 = m\|L_m\|\omega(y,h)/\|y\|$.

Thus we obtain the analogue of theorem 2.3.4, showing the same conditions on the kernel are sufficient for the Kantorovich convergence theorem (theorem 1.3.3).

Theorem 3.6.3

If the integral operator, K, on C is approximated by a composite, interpolatory quadrature rule, then it is sufficient for the application of the Kantorovich convergence theorem in the space C that the two partial moduli of continuity of the kernel exist as simple moduli of continuity.

Proof:

By lemmas 3.6.1 and 3.6.2 conditions I, II, III are satisfied for all n. Since the equation in C implies $y\epsilon C$, the hypotheses on the partial moduli of continuity imply $\bar{\eta}$, η_1, $\eta_2 \to 0$ as $n\to\infty$. Since $\tilde{K}_n$ has a finite-dimensional range it is a compact operator and hence the second hypothesis of theorem 1.3.3 is satisfied. Hence theorem 1.3.3 applies if the equation has a unique solution.

As in section 2.3, the continuity of the kernel k implies the required conditions on the partial moduli of continuity. Hence, by theorem 1.3.3, the assumption of the existence of $(I-K)^{-1}$ implies the algebraic system (3.6.1) is soluble for sufficiently large n (as in the Anselone theory), and the approximate solutions $\varphi_o^{-1}x^*$ (the piecewise

polynomial functions obtained from the solution of (3.6.1)) converge to the exact solution.

The condition III is not generally required in Anselone's theory since the approximate equation has the same right side as the given equation. However the condition is in fact used by Anselone and Gonzalez-Fernandez [6, p. 256] in the case of the iterated operator equation where the new right side would become $u = (I+K+\ldots+K^{p-1})y$. If u is obtained exactly there is no problem but ordinarily u would also be calculated by approximate integration, yielding an approximate value u_n.

In the case of Anselone's theory the approximate equation is also considered in the space X, so that $X = \tilde{X}$ and conditions II and III are trivially satisfied. Condition I becomes the condition that K and $\tilde{K}$ are "close in the space X": $\|K-\tilde{K}\|<\eta$. If K_n were an integral operator with a kernel $k_n(s,t)$ such that $k_n(s,t)\to k(s,t)$ uniformly as $n\to\infty$, then $\|K_n-K\|\to 0$, and the standard theory, based on operator norm convergence, for the convergence of the approximate solutions would follow (Proposition 1.3.1).

However operator norm convergence does not hold where the operator approximation is based on a numerical quadrature formula since in this case [1,3] $\|K_n-K\| \geq \|K\|$, $n\geq 1$. In the space X condition Ia becomes (compare (3.6.2))

$$\|\varphi Kx-\bar{K}\varphi x\| \leq \max_j \int_0^1 |k_{t_j} x - P_n^m k_{t_j} x|\,dt$$

$$\leq [C(\omega_t(h)+\omega(x,h)\cdot\|k\|/\|x\|)]\,\|x\| \qquad \text{by (2.3.1)}$$

The term in brackets may be taken as $\bar{\eta}$, but in this case $\bar{\eta}$ clearly depends on x. Thus, as with Anselone's theory, we have only pointwise convergence

of the approximate operators on X. Anselone compensates for this inadequacy by showing that compactness of the operator gives uniform convergence for a certain approximation of the iterated operator.

The primary difference between the two theories is in the interpolation of the discrete algebraic system (3.6.1) to form an approximate solution. Kantorovich interpolates the solution of (3.6.1) into the approximate space $\tilde{X}$, while Anselone uses the approximate equation to interpolate it into the space X. However, we have shown that for integral equations in the space C, the same conditions are sufficient to apply either theory (Theorem 2.3.4 and Theorem 3.6.3).

Finally, considering Gabdulhaev's method again, from the point of view of the Kantorovich theory, it is noted that condition I (equivalently Ia) is obviously satisfied with $\eta=0$ since the operator $\tilde{K}$ is taken to be PK. Thus the approximate operators converge in the approximate space $\tilde{H}(\beta)$, but as noted previously, this does not imply convergence in $H(\beta)$. We still require $\eta_1\|P\|$, $\eta_2\|P\|\to 0$ for the convergence of the approximate solutions in Theorem 1.3.3. The Kantorovich theory states that these conditions are satisfied in this case if $P_n\to I$ on the complete space $H(\beta)$ and the operator U is compact [25, Theorem 6, p. 555]. But it was shown (lemma 3.3.5) that it is not true that $P_n\to I$ on $H(\beta)$. For condition II we have

$$\|Ux-\tilde{x}\|_\beta = \|Ux-PUx\|_\beta \le 3\sup_{0<\delta\le h} \omega(Ux,\delta)/\delta^\beta \text{ by lemma 3.3.3 with } m=1,$$

and $x\in H(\beta)$ implies $Ux\in H(\beta)$. We can assert only that the right side is bounded and not that it converges. Under Gabdulhaev's conditions (3.5.1) and the assumption $x\in H(\delta)$, $0<\beta<\delta<\alpha$, then $Ux\in H(\delta)$ and the right side is bounded by $3\|U\|_\beta\, M_2[x;\delta]\,h^{\delta-\beta}$. Hence condition II is formally satisfied with a term

similar to that in the convergence condition (3.5.4). That is Gabdulhaev has effectively assumed a "compactness" in condition for the operator U, but as noted in section 3.5.1, this is not valid since we are not justified in restricting x to the space $H(\delta)$.

It is concluded that neither Kantorovich's theory, nor Anselone's theory, may be applied to justify Gabdulhaev's method of direct approximation of the singular integral equation.

3.7 Concluding Remarks

It has been shown by the analysis of Gabdulhaev's paper that an attempt to ensure uniform convergence by considering the metric of a coarser topology, $H(\beta)$, for the space of functions $H(\alpha)$, $(\beta<\alpha)$, fails basically because the space with this metric is no longer complete. Equivalently an attempt to seek a solution in a "smoother" space of functions fails for the same reason. The introduction of a sufficiently smooth function k into the kernel, $k(s,t)/(t-s)$, does not help since for $x \in H(\beta)$, Kx still belongs to $H(\beta)$ (section 3.4.3). This is not unexpected since an operator of this form readily reduces to the sum of a Fredholm operator and a Hilbert operator.

The only possibility that would be of any assistance in this analysis is to ensure that the kernel is absolutely integrable. But this essentially implies that $k(s,t)$ has a zero on the diagonal $s=t$, and thus reduces to the case of a Fredholm operator with a weakly singular kernel. This case was considered in section 2.5.

It was noted in Chapter 2 that for a Fredholm equation with given functions in C, every bounded, integrable solution also lies in C. Thus we may seek approximate solutions which converge uniformly (in the sup norm) to the exact solution. If the right side of the equation is continuously

differentiable and $\int \left|\frac{\partial k(s,t)}{\partial s}\right| dt = g'(s) \in C$, then the equation lies in the smoother space C'. A similar argument shows that every bounded, integrable solution of this equation necessarily lies in the required space C'. Thus it is again sufficient to require convergence of the approximate solutions in the sup norm, providing the approximate operators converge pointwise in the C' norm. That is, by analogy with lemma 2.3.1, providing the set $\{(\frac{\partial k(s,t)}{\partial s})_s : 0 \leq s \leq 1\}$ is equicontinuous. The same argument applies in the case of a Fredholm equation in the space H(β) (if $\int |k(s,t)| dt = g(s) \in H(\beta)$). It follows from lemma 3.4.3 that approximate solutions converging in the sup norm also converge in the H(β) norm providing the approximate operators converge in H(β), pointwise for $x \in R$ (i.e., if $\{k_s\}$, $\{k_t\}$ are both equi-H(β)).

However this argument cannot be applied to a singular operator since the principal value integral is not absolutely integrable. It was noted in Section 3.4.3 that it is necessary that the operand also satisfies some Hölder condition, implying that a direct approximation requires convergence in the Hölder space.

Thus it is concluded that the Hilbert operator is not sufficiently "smoothing" to permit direct application of the uniform approximation theory. This would include a scheme involving perturbation of the kernels (compare possibilities in Titchmarsh [45]) since the required properties ultimately depend on the compactness of the original operator (compare Anselone [4]). Similarly a sequence of operator differences still does not avoid the basic problem that the Cauchy principal value integral is only conditionally and not absolutely convergent.

Hence we finally conclude that the only practical way to deal numerically with a singular integral equation is to consider some semi-indirect method in which the singular equation is first reduced to the more tractable case of an equation with a Fredholm operator.

4. UNIFORM APPROXIMATION OF THE SOLUTION OF SINGULAR INTEGRAL EQUATIONS IN A HÖLDER SPACE

4.1 Introduction

It has been shown in the preceding chapters that a singular integral equation is not amenable to a direct method of uniform approximation. The method proposed in this chapter might be classed as a semi-direct method since the equation is first reduced analytically to the case of a Fredholm equation. In principle the latter equation can be dealt with directly according to the theory in Chapter 2. However, a practical approximation scheme involving only the given functions (section 4.3) requires some additional analysis to show that the approximate solution of the equivalent Fredholm equation does in fact converge uniformly to the desired solution.

It is shown that it is necessary to strengthen the hypothesis on the density function $k(s,t)$ so that this function is required to satisfy a Hölder condition with exponent $\nu>\mu$ with respect to both variables, (instead of exponent μ as in Muskhelishvili and Pogorzelski) in order to obtain a consistent equation in the space $H(\mu)$. The forms of the reduced Fredholm equations are given and an analysis of the convergence of an approximation to the weakly singular equation obtained by Noether's method is given.

4.2 The Singular Integral Equation

To avoid the complications at the end-points of a non-closed arc, we consider the range of integration to be a smooth closed Jordan contour L (compare [33, Chapter 6]). Consider the linear singular integral equation of the normal [38] type:

$$M\varphi(s) \equiv (aI+K)\varphi(s) = a(s)\varphi(s) + \frac{1}{\pi i}\int_L \frac{k(s,t)\varphi(t)}{t-s}dt = f(s) \qquad (4.2.1)$$

where a,k,f are given functions on L belonging to some Hölder class $H(\mu)$.

Muskhelishvili [33, p. 114] seeks the solution of this equation which also satisfies the "H condition"; that is, the Hölder condition for some unspecified exponent. It is implied indirectly [33, p. 132] that the solution belongs to $H(\mu')$, $0<\mu'<\mu$ (compare also [17, p. 218] and section 3.5.1). The solution of the Hilbert problem belongs to the same Hölder class as the given functions [38, p. 466] and thus the solution of the equivalent dominant equation [38] belongs to $H(\mu)$ if the given functions belong to $H(\mu)$, $\mu<1$.

For the complete singular integral equation (4.2.1) the solution space is less clear. Despite the fact that some authors (Dolph [15, p. 152]) apparently infer that the preservation of the Hölder class, for a density function of two variables, follows as an immediate consequence of the Privalov theorem, the generalized Privalov theorem [38, p. 443] implies a weaker result (see discussion at the beginning of section 3.4.3). Thus $K:H(\mu)\rightarrow H(\mu)$ if $k\in H(\nu,\mu)$, $\nu>\mu$, but not if $k\in H(\mu)$.

The theoretical investigation and solution of the complete linear singular integral equation is obtained by reducing the equation to an equivalent Fredholm (or regular) equation. Any of the three methods of regularization (or reduction) in use [18, Ch. 3] ultimately involve the product of the singular operator with itself. The expression for the product operator, using the Poincaré-Bertrand transformation (section 3.1) involves a kernel of the form

$$\int_L \frac{k(s,u)k(u,t)}{(u-s)(t-u)}du = \frac{1}{t-s}\left\{\int_L \frac{k(s,u)k(u,t)}{u-s}du - \int_L \frac{k(s,u)k(u,t)}{u-t}du\right\}$$

By the generalized Privalov theorem, the first integral on the right belongs to $H(\mu,\mu')$ and the second integral belongs to $H(\nu',\mu')$ where μ', ν' are arbitrary positive constants less than μ,ν respectively. The resulting density function thus satisfies the $H(\mu,\mu')$ class, choosing $\mu'<\mu<\nu'<\nu$. Hence the corresponding product operator maps $H(\mu)\to H(\mu')$ (by the generalized Privalov theorem).

Hence in order that the reduced equation is also an equation in the space $H(\mu)$ we require that the density function, k, satisfy $k\in H(\nu)$, $\nu>\mu$. With this change in hypothesis in equation (4.2.1) we have an equation in the space $H(\mu)$ and may seek a solution in this space.

The following lemma concerning the associated weakly singular operator L is also noted.

Lemma 4.2.1 [38, lemma 2, p. 495]

If $k(s,t)\in H(\nu)$ then the weakly singular operator, L, with kernel $\ell(s,t) = (k(s,t)-k(s,s))/(t-s)$, transforms each bounded, integrable function, φ, into the space $H(\mu)$ for $\mu<\nu$.

Actually Pogorzelski's result states that the transformed function satisfies the condition with exponent $\nu/2$. But an analysis of this proof and the corresponding result of Muskhelishvili [33, p. 135] reveals that in effect it is shown that $L\varphi\in H(\lambda)$ where $\lambda = \min(\nu-\alpha, 1-\alpha)$, where α is an arbitrary positive number less than ν. Thus since $0<\nu\leq 1$, setting $\alpha = \nu-\mu$ gives the desired result.

4.2.1 Reduction of the Singular Integral Equation

The theory of reducing operators may be found in Muskhelishvili [33] and Pogorzelski [38].

The operator adjoint to the dominant operator, M^{o}, is defined by

$$M^{o'}\psi(s) \equiv a(s)\psi(s) - \frac{1}{\pi i}\int_L \frac{b(t)\psi(t)}{t-s}dt \quad , \text{ where } b(t) = k(t,t). \qquad (4.2.2)$$

Noether's method uses the operator $M^{o'}$ as the reducing operator. Equation (4.2.1) is reduced to the weakly singular Fredholm equation

$$M^{o'}M\varphi(s) \equiv N\varphi(s) \equiv (a^2(s)-b^2(s))\varphi(s) + \frac{1}{\pi^2}\int_L \{\int_L \frac{b(u)k(u,t)}{(u-s)(t-u)}du\}\varphi(t)dt = M^{o'}f \qquad (4.2.3)$$

The kernel of the integral operator may be written

$$n(s,t) = \int_L \frac{b(u)k(u,t)}{(u-s)(t-u)}du = \frac{1}{t-s}\{\int_L \frac{b(u)k(u,t)}{u-s}du - \int_L \frac{b(u)k(u,t)}{u-t}du\} = \frac{q(s,t)-q(t,t)}{t-s}$$

where $q(s,t)\in H(\nu,\nu')$ by the generalized Privalov theorem.

Although solutions of the Fredholm equation are normally sought in the class of continuous functions, there is no necessity to impose beforehand the additional condition that the solution of (4.2.3) belongs to the space $H(\mu)$ since, by lemma 4.2.1, each bounded, integrable solution of (4.2.3) necessarily belongs to $H(\mu)$.

The index, κ, of M is defined by the integer $\kappa \equiv \frac{1}{2\pi i}[\log\frac{a-b}{a+b}]_L$ where the symbol $[]_L$ stands for the increment, suffered by the function in braces, on one circuit of L in the positive direction. The theory of Noether's method shows that in the case that the index, κ, of M is non-negative, the reduced equation is equivalent to the original equation. If $\kappa<0$ then it is impossible to find a reducing operator of this type.

However, Vekua's theorem of equivalence [38, p. 512] shows how to find an equivalent Fredholm equation in the case the index is negative. Introduce the new unknown function, ψ, into the singular equation by the

substitution $\varphi(s) = M^{o'}\psi(s)$. Then, for $\varkappa<0$, the equation (4.2.1) is reduced to the weakly singular Fredholm equation

$$M(M^{o'}\psi)(s) \equiv (a^2(s)-b^2(s))\psi(s) + \frac{1}{\pi^2}\int_L\left\{\int_L \frac{k(s,u)b(t)}{(u-s)(t-u)}du\right\}\psi(t)dt = f(s) \qquad (4.2.4)$$

As before the determination of the solutions, φ, of the original equation (4.2.1) is equivalent to the determination of the solutions, ψ, of (4.2.4), where φ is obtained by the additional step of evaluating $\varphi = M^{o'}\psi$.

Hence, after the determination of the sign of the index of the equation, the singular integral equation may be approximated by considering the appropriate form (4.2.3) or (4.2.4).

The singular integral equation may not have a unique solution (the number of linearly independent solutions is related to the index). The approximation theory (section 1.3.2) assumes that the given equation has a unique solution. It is noted that the problem of obtaining a particular solution of a Fredholm equation with a non-unique solution has been considered by Atkinson [8], who obtains a selective inverse. A similar approach may yield particular solutions in the case of a singular integral equation, but this problem was not investigated.

4.3 Approximation and Convergence

The preliminary analysis reducing the singular equation to a Fredholm equation before the application of approximate methods seems to be an unfortunate but necessary condition. On the other hand this reduction does allow the approximation to be considered in the space C which has a much simpler norm than the Hölder norms. Thus the complications of the determination of the appropriate Hölder space and calculations with the

relatively unwieldy Hölder norms are avoided. Further for a given singular equation, the appropriate values may be substituted directly into the approximate equations obtained in this section and so for practical purposes the details of the reduction process may be ignored. The only preliminary calculation that is necessary is the determination of the sign of the index of the equation in order to decide which of the forms ((4.2.3) or (4.2.4)) of the Fredholm equation it is necessary to consider. It is noted that Gakhov [18, p. 89] gives a computational method of determining the index.

For the purposes of this section we suppose a singular integral equation (4.2.1) is given, that it has a unique solution, and that it has a non-negative index, so that its approximate equation will be determined from equation (4.2.3). Equation (4.2.3) may be written (compare section 4.2.1)

$$N\varphi \equiv [(a^2-b^2)I + \frac{1}{\pi^2}Q]\varphi = g \qquad (4.3.1)$$

where I is the identity operator,

$$Q\varphi(s) = \int_L \frac{q(s,t)-q(t,t)}{t-s}\varphi(t)dt,$$

and

$$g(s) = M^{o'}f(s) = a(s)f(s) - \frac{1}{\pi i}\int_L \frac{b(t)f(t)}{t-s}dt$$

Since the kernel of the operator Q is weakly singular an immediate approach might utilize the approximate product integration technique of Atkinson [16]. However in addition to the difficulty of determining a suitable split in the integrand it is noted that the functions q are themselves expressed as princi- pal value integrals. Thus, in general, we will also have q replaced by some approximation q_n.

In order to simplify the arguments, we will assume here that the operator P_n^t represents piecewise linear interpolation with respect to the variable t, but, as noted in Chapter 2, interpolation by higher order polynomials involves only a change in the constants and not in the substance of the argument. Define an approximate operator Q_n by

$$Q_n\varphi(s) = \int_L \frac{P_n^t[(q(s,t)-q(t,t)\varphi(t)]}{t-s}dt \tag{4.3.2}$$

and an approximate operator Q_n^n by

$$Q_n^n\varphi(s) = \int_L \frac{P_n^t[(q_n(s,t)-q_n(t,t))\varphi(t)]}{t-s}dt \tag{4.3.3}$$

where q_n is some approximation to q.

Defining the approximate operator $N_n \equiv (a^2-b^2)I + \frac{1}{\pi^2}Q_n^n$, it is necessary to obtain pointwise convergence in the space C, of the operators N_n (and thus of Q_n^n) for the application of the abstract theory. Thus it is required to show that $Q_n \to Q$ pointwise, and, since this result depends on the Hölder properties of q, $q_n \to q$ uniformly in the Holder norm of $H(\mu)$.

Define the approximation $q_n(s,t)$ by

$$q_n(s,t) = \int_L \frac{P_n^u(b(u)k(u,t))}{u-s}du \tag{4.3.4}$$

<u>Lemma 4.3.1</u>

Let q_n be defined by (4.3.4). Then $q_n \to q$ as $n\to\infty$, uniformly with respect to s,t in the $H(\mu)$ norm where $k(u,t)\in H(\nu)$, $\nu>\mu$.

Proof:

Following the analysis in Chapter 3 and the generalized Privalov theorem we have

$$M_1[q_n(s,t)-q(s,t)] \le \pi\, M_1[P_n^u b(u)k_t(u)-b(u)k_t(u)]$$
$$+ C'M_2[(P_n^u b(u)k_t(u)-b(u)k_t(u));\mu]$$
$$\le \pi\, M_2[bk_t;\nu]\,h^\nu + 2C'M_2[bk_t;\nu]\,h^{\nu-\mu}$$

using the relations (3.3.1) and (3.3.7) for m=1. Note that these Holder constants exist finitely since $b(u) = k(u,u)\varepsilon H(\nu)$, and that $h = 0(\frac{1}{n})$.

Let ℓ be the finite length of the arc L and ν' an arbitrary value satisfying $\mu<\nu'<\nu$. Then from the generalized Privalov theorem again

$$M_2[(q_n(s,t)-q(s,t));\mu] \le C_2M_2[(P_n^u b(u)k_t(u)-b(u)k_t(u));\mu]$$
$$+ \sup_{0<|t_1-t_2|<\ell}\Big\{\pi M_2[(P_n^u b(u)k_t(u)-b(u)k_t(u));\nu']\,|t_1-t_2|^{\nu'-\mu}$$
$$+ C_3M_2[(P_n^u b(u)k_t(u)-b(u)k_t(u));\nu']\,|t_1-t_2|^{\nu'-\mu}$$
$$\cdot \log|t_1-t_2|\Big\}$$

where C_2, C_3 are constants. Then again applying the estimate

$$M_2(P_nx-x;\mu) \le 2M_2(x;\nu)h^{\nu-\mu} \text{ for } x\varepsilon H(\nu), \text{ (relation (3.3.7))}$$

we obtain the inequality

$$M_2[(q_n(s,t)-q(s,t));\mu] \leq C_4 M_2[bk_t;\nu]\, h^{\nu-\nu'}, \quad C_4 \text{ constant.}$$

Combining these results

$$\|q_n(s,t)-q(s,t)\|_\mu = M_1[q_n(s,t)-q(s,t)] + M_2[(q_n(s,t)-q(s,t));\mu]$$

$$\leq C_5\, M_2[bk_t;\nu]\, h^{\nu-\nu'} \tag{4.3.5}$$

Since C_5 is a constant, the Holder constants are uniform in u and t, and $h = 0(\frac{1}{n})$, the relation (4.3.5) gives the required convergence as $n\to\infty$.

A similar result of course holds for the function q(t,t).

Lemma 4.3.2

For each fixed $\varphi\epsilon H(\mu)$ the functions $Q_n\varphi(s)$ obtained from (4.3.2) converge to $Q\varphi(s)$ in the norm of the space C.

Proof:

Let the sup norm be denoted by the subscript C. Let μ' be a positive constant satisfying $0<\mu'<\mu$. Then by an argument similar to that used in obtaining (4.3.5) in the previous lemma we obtain

$$\|(Q_n-Q)\varphi\|_C = M_1[(Q_n-Q)\varphi]$$

$$\leq B\,\{M_1[P_n^t(q(s,t)-q(t,t))\varphi(t)-(q(s,t)-q(t,t))\varphi(t)]$$

$$+ M_2[(P_n^t(q(s,t)-q(t,t))\varphi(t)-(q(s,t)-q(t,t))\varphi(t));\mu']\}$$

where B is a constant. Since $q\epsilon H(\nu')\subset H(\mu)$ and $\varphi\epsilon H(\mu)$, the usual estimates on these terms give

$$\|(Q_n-Q)\varphi\|_C \le B_1 \cdot M_2[(q(s,t)-q(t,t))\varphi(t)\ ;\mu]\, h^{\mu-\mu'} \tag{4.3.6}$$

and the required convergence is obtained as $n\to\infty$ since $h = 0(\frac{1}{n})$, B is a constant, and the Holder constant is uniform in s and t.

The results of these two lemmas are now combined to obtain the required pointwise convergence for the approximate operators Q_n^n.

Lemma 4.3.3

For each fixed $\varphi\epsilon H(\mu)$ the functions $Q_n^n\varphi(s)$ obtained from (4.3.3) converge to $Q\varphi(s)$ in the norm of the space C.

Proof:

From the identity $Q_n^n\varphi-Q\varphi = Q_n^n\varphi-Q_n\varphi + Q_n\varphi-Q\varphi$ we obtain

$$\|Q_n^n\varphi-Q\varphi\|_C \le \|(Q_n^n-Q_n)\varphi\|_C + \|Q_n\varphi-Q\varphi\|_C$$

The second term converges by lemma 4.3.2. Arguing as in the previous lemma

$$\begin{aligned}\|(Q_n^n-Q_n)\varphi\|_C &= M_1[(Q_n^n-Q_n)\varphi] \\ &\le B\|P_n^t[((q_n(s,t)-q_n(t,t))-(q(s,t)-q(t,t)))\varphi(t)]\|_\mu \\ &\le B.\|P_n^t\|_\mu\|\varphi\|_\mu\{\|q_n(s,t)-q(s,t)\|_\mu+\|q_n(t,t)-q(t,t)\|_\mu\} \qquad (4.3.7)\end{aligned}$$

$\|\varphi\|_\mu$ is a constant for a fixed $\varphi\epsilon H(\mu)$ and it is easily shown (compare Gabdulhaev [42, p. 215]) that $\|P_n^t\|_\mu \le 3$. Thus we have convergence as $n\to\infty$ by lemma 4.3.1.

Actually in view of (4.3.7) and (4.3.6) the convergence in the above lemma is uniform over a bounded set of functions $\varphi \in H(\mu)$ but such a result is not surprising in view of the fact that Q is a weakly singular Fredholm operator and a bounded set in $H(\mu)$ is equicontinuous in C.

By lemma 4.2.1, the set $\{Q\varphi : \|\varphi\|_C \leq 1\}$ is Holder continuous (since the kernel is weakly singular) and thus is a bounded, equicontinuous set in the space C. That is Q is a compact operator in the space C. In fact we can make the stronger statement that Q is a compact operator in the space $H(\mu)$.

Lemma 4.3.4

The weakly singular operator Q defined in (4.3.1) is compact in the space $H(\mu)$, $\mu<1$.

Proof:

Let B be a bounded set of functions in $H(\mu)$. By definition the operator Q is compact if the set QB is precompact and this latter condition holds if, from any sequence $\{Q\varphi_n\}$ of its points, we can choose a subsequence $\{Q\varphi_{n_k}\}$ which is convergent in the $H(\mu)$ norm. The functions of the sequence $\{\varphi_n\} \subset B$ are equicontinuous and uniformly bounded and hence, by Arzela's theorem, we may select a subsequence $\{\varphi_{n_k}\}$ which is uniformly convergent (in the usual sense of the sup norm). It follows that for each $\epsilon>0$, there exists N_ϵ such that

$$\sup_t |\varphi_{n_j}(t)-\varphi_{n_k}(t)| < \epsilon \text{ holds for } j,k > N_\epsilon$$

Thus $$M_1[Q\varphi_{n_j}-Q\varphi_{n_k}] = \max_s \left|\int_L \frac{q(s,t)-q(t,t)}{t-s}(\varphi_{n_j}(t)-\varphi_{n_k}(t))dt\right|$$

$$\leq \|\varphi_{n_j}-\varphi_{n_k}\|_C \cdot M_2[q_t;\nu] \cdot \max_s \left|\int_L |t-s|^{\nu-1}|dt|\right|$$

$$\leq C_1 \cdot M_2[q_t;\nu] \cdot \epsilon \text{ for } j,k>N_\epsilon,\ C_1 \text{ constant.} \tag{4.3.8}$$

and $M_2[(Q\varphi_{n_j} - Q\varphi_{n_k});\mu]$

$$= \sup_{\substack{0<|s_1-s_2|<\ell \\ s_1,s_2\in L}} \left| \frac{1}{|s_1-s_2|^{\mu}} \left\{ \int_L \frac{(q(s_1,t)-q(t,t))-(q(s_2,t)-q(t,t))}{t-s_1} (\varphi_{n_j}(t)-\varphi_{n_k}(t))dt \right.\right.$$

$$\left.\left. + \int_L (q(s_2,t)-q(t,t))(\varphi_{n_j}(t)-\varphi_{n_k}(t))\left(\frac{1}{t-s_1} - \frac{1}{t-s_2}\right)dt \right\} \right|$$

By the generalized Privalov Theorem and an analysis similar to that used in the proof of lemma 4.3.1, the first term is bounded by

$$C_2 \cdot \|\varphi_{n_j} - \varphi_{n_k}\|_C \cdot M_2[q_t;\nu] \cdot |s_1-s_2|^{\nu-\mu}(1+\log|s_1-s_2|)$$

$$\leq C_3 \cdot M_2[q_t;\nu] \cdot \varepsilon \qquad (4.3.9)$$

for some constant C_3, and $j,k > N_\varepsilon$, where $\nu > \mu$.

By the same argument as used in the proof of the Privalov theorem the second term is bounded by

$$C_4 \cdot \|\varphi_{n_j} - \varphi_{n_k}\|_C \cdot M_2[q_t;\nu] \cdot |s_1-s_2|^{\nu-\mu} \qquad \text{for } \nu<1$$

$$\leq C_5 \cdot M_2[q_t;\nu] \cdot \varepsilon \qquad \text{for } j,k > N_\varepsilon \qquad (4.3.10)$$

Hence for $\mu<1$ we conclude from (4.3.8), (4.3.9), (4.3.10) that the sequence $\{Q\varphi_{n_k}\}$ satisfies the Cauchy criterion for convergence in the $H(\mu)$ norm:

$$\|Q\varphi_{n_j}-Q\varphi_{n_k}\|_\mu = M_1[Q\varphi_{n_j}-Q\varphi_{n_k}] + M_2[(Q\varphi_{n_j}-Q\varphi_{n_k});\mu]$$

$$\leq C_6 \cdot M_2[q_t;\nu] \cdot \varepsilon \qquad \text{for } j,k > N_\varepsilon$$

Since the space $H(\mu)$ is complete the Cauchy criterion is sufficient for the convergence of the sequence of points and we conclude that $\{Q\varphi_{n_k}\}$ is convergent in the $H(\mu)$ norm.

Note that the result may also be obtained using corollary 3.3.2 since the above proof also shows that the operator Q transforms a bounded set into a bounded set in $H(\nu')$ which is compact in $H(\mu)$. The result may also be obtained more directly by expressing the kernel in the form $\frac{q(s,t)-q(t,t)}{t-s} = \frac{q^*(s,t)}{|t-s|^{1-\alpha}}$ where α is an arbitrary positive number less than ν' and $q^*(s,t)\in H(\nu'-\alpha)$ [38, lemma 1, p. 492] and using lemma 4.2.1. However this approach is not amenable to obtaining a similar result for the collective compactness of the operator approximations as is suggested by theorem 2.3.3. Note that the proof of the above lemma gives:

$\sup_t \omega(q_t,\delta)/\delta^\mu \to 0$ as $\delta \to 0$ implies Q is compact in $H(\mu)$, analogous to the result of theorem 2.3.2, and thus we may expect the same condition to also give $\{Q_n^n : n\geq 1\}$ is collectively compact.

<u>Lemma 4.3.5</u>

The approximate operators $\{Q_n^n : n\geq 1\}$ defined by (4.3.3) are collectively compact in the space $H(\mu)$, $\mu<1$

<u>Proof</u>:

This set is collectively compact by definition if the set $\{Q_n^n B : n\geq 1\}$ is compact. Proceeding as in the proof of the preceding lemma, let $\{\varphi_{m_k}\}$ be a uniformly convergent (in C) subsequence obtained from the set B by Arzela's theorem. Thus

$$M_1[Q_n^n\varphi_{m_j} - Q_n^n\varphi_{m_k}] = \max_s \left| \int_L \frac{P_n[(q_n(s,t)-q_n(t,t))(\varphi_{m_j}(t)-\varphi_{m_k}(t))]\,dt}{t-s} \right|$$

$$\leq \|P_n\|_C \cdot \|\varphi_{m_j} - \varphi_{m_k}\|_C \cdot M_2[q_{nt};\nu] \,.\max_s \int_L |t-s|^{\nu-1} |dt|$$

$$\leq C_1 \cdot M_2[q_{nt};\nu] \,.\varepsilon \qquad \text{for } j,k > M_\varepsilon,\ C_1 \text{ constant.}$$

Note that $\|P_n\| = 1$ for linear interpolation but the argument would not be affected if P_n was piecewise interpolation by m^{th} degree polynomials since then $\|P_n\| = \|L_m\|$ where L_m is the Lagrange operator. Similarly, following the proof of lemma 4.3.4,

$$M_2[(Q_n^n\varphi_{m_j} - Q_n^n\varphi_{m_k});\mu] \leq C_2 M_2[q_{n_t};\nu] \,.\varepsilon \qquad \text{for } j,k > M_\varepsilon$$

Hence the result will be established if we can bound $M_2[q_{nt};\nu]$ uniformly with respect to n.

$$q_n(s_1,t) - q_n(s_2,t) = \int_L P_n^n b(u)k(u,t)\left(\frac{1}{u-s_1} - \frac{1}{u-s_2}\right)du$$

$$= \pi(P_n bk_t(s_1) - P_n bk_t(s_2)) + \Phi(s_1,t) - \Phi(s_2,t) \qquad (4.3.11)$$

where

$$\Phi(s,t) = \int_L \frac{P_n bk_t(u) - P_n bk_t(s)}{u-s}\,du$$

and $P_n bk_t(s)$ represents the piecewise linear interpolation of the function bk_t, evaluated at the point s.

Suppose $x\epsilon H(\mu)$. Then $P_n x \epsilon H(1) \subset H(\mu)$. Thus

$$|P_n x(s_1)-P_n x(s_2)| \le \omega(P_n x,|s_1-s_2|) \le M_2[P_n x;\mu].|s_1-s_2|^{\mu}$$

Further $M_2[P_n x;\mu] \le M_2[x;\mu] + M_2[P_n x-x;\mu] \le 3M_2[x;\mu]$

Thus $$|P_n x(s_1)-P_n x(s_2)| \le 3\, M_2[x;\mu].|s_1-s_2|^{\mu}.$$

Applying this result to (4.3.11) and applying techniques similar to those in the proof of the Privalov theorem in the estimation of the integral term, Φ, we obtain (compare lemma 3.4.4)

$$\begin{aligned}|q_n(s_1,t)-q_n(s_2,t)| &\le \sup_t\{3\pi M_2[bk_t;\nu]\,|s_1-s_2|^{\nu} + 3\, C_2 M_2[bk_t;\nu].|s_1-s_2|\} \\ &\le C_3\, M_2[k(s,s)k(s,t);\nu]\,|s_1-s_2|^{\nu}\end{aligned}$$

Thus, since $k\epsilon H(\nu)$,

$$M_2[q_{nt};\nu] \le 2C_3 M_1[k] M_2[k;\nu] \quad , \text{ independent of } n.$$

Hence

$$\begin{aligned}\|Q_n^n\varphi_{m_j}-Q_n^n\varphi_{m_k}\|_{\mu} &= M_1[Q_n^n\varphi_{m_j}-Q_n^n\varphi_{m_k}] + M_2[(Q_n^n\varphi_{m_j}-Q_n^n\varphi_{m_k});\mu] \\ &\le C_4.M_1[k].M_2[k;\nu].\varepsilon \qquad \text{for } j,k > M_{\epsilon}\end{aligned}$$

and it follows that the set $\{Q_n^n : n\ge 1\}$ is collectively compact since $\{Q_n^n\varphi : \|\varphi\|_{\mu}\le 1, n\ge 1\}$ is compact in $H(\mu)$ by the previous lemma.

Since compactness in $H(\mu)$ implies compactness in the coarser topology of the space C, the lemmas 4.3.3, 4.3.4 and 4.3.5 show that the hypotheses of the convergence theorem 1.3.2 are satisfied and thus the solution, φ_n, of the approximate equation, $N_n\varphi_n = g$, converges to the exact solution, φ, of equation (4.3.1) in the norm of the space C. In view of lemma 4.2.1, this limit of this sequence of approximate solutions satisfies the required Hölder condition and thus is a solution of the equivalent singular integral equation (4.2.1).

However because of the nature of the right side, g, it would be more practical to consider the approximate equation

$$N_n\varphi_n = g_n \tag{4.3.12}$$

where g_n is some approximation to the right side, $g = M^{o'}f$, of (4.3.1). Let the approximation g_n be obtained in the usual way of piecewise linear interpolation of the numerator of the integrand.

<u>Lemma 4.3.6</u>

The approximation g_n converges uniformly to g.

<u>Proof:</u>

$$\begin{aligned}\|g_n-g\|_C &= \left\|af - \frac{1}{\pi i}\int_L \frac{P_n bf(t)}{t-s}dt - af + \frac{1}{\pi i}\int_L \frac{b(t)f(t)}{t-s}dt\right\|_C \\ &\le B\|P_n bf(t)-b(t)f(t)\|_{\mu/2} \qquad \text{(compare lemma 4.3.3)} \\ &\le 3B\, M_2[bf;\mu]\, h^{\mu/2} \quad \to 0 \text{ as } n\to\infty, \quad \text{since } b\in H(\nu)\subset H(\mu) \text{ and}\end{aligned}$$

$f\in H(\mu)$ implies $M_2[bf;\mu]$ is a constant.

The collectively compact operator approximation theory implies that if N^{-1} exists then N_n^{-1} exists for sufficiently large values of n and

conversely. Thus we may assume that both N^{-1} and N_n^{-1} exist and further that N_n^{-1} is bounded uniformly in n for sufficiently large n. Hence we have the convergence theorem:

Theorem 4.3.7

The solution, φ_n, of the approximate equation (4.3.12) converges to the solution, φ, of the singular integral equation (4.2.1)

Proof:

From (4.3.1) and (4.3.12)

$$\begin{aligned}\varphi_n-\varphi &= N_n^{-1}g_n-\varphi\\ &= N_n^{-1}(g_n-N_n\varphi)\\ &= N_n^{-1}(g_n-g+N\varphi-N_n\varphi)\end{aligned}$$

Hence

$$\begin{aligned}\|\varphi_n-\varphi\|_C &\le \|N_n^{-1}\|_C\{\|g_n-g\|_C + \|(N-N_n)\varphi\|_C\}\\ &\le B\{\|g_n-g\|_C + \|(Q-Q_n^n)\varphi\|_C\} \qquad (4.3.13)\end{aligned}$$

$\to 0$ as $n\to\infty$ by lemmas 4.3.3 and 4.3.6, where B is the uniform bound for N_n^{-1}.

Thus $\{\varphi_n\}$ converges to the solution of (4.3.1), has the required Holder property by lemma 4.2.1, and thus converges also to the solution of the equivalent singular integral equation (4.2.1) by the equivalence theorem (section 4.2.1). Since $q\epsilon H(\nu,\nu')$ the sets $\{q_t\}$,$\{q_s\}$ are both equi-$H(\mu)$ and thus convergence of the approximate solutions in $H(\mu)$ follows as in lemma 3.4.3.

Note that the general theory gives computable error bounds for $\|\varphi_n-\varphi\|$. These are normally obtained from the error formula for the particular quadrature rule but it has been shown in the preceding lemmas that the error

bounds are considerably more complex in this case. Bounds for (4.3.13) could be obtained from the estimates in lemmas 4.3.6, 4.3.2 and 4.3.1 but it would not seem to be practical to try to estimate these constants. At best the order of convergence could be estimated as $O(h^{\lambda})$ where $\lambda = \min(\mu/2, \nu-\nu')$ thus convergence would be expected to be slow and some method of accelerating convergence would seem advisable.

As usual in the general approximation theory discussed previously, the approximate equation (4.3.12) is not solved directly but rather the equivalent algebraic linear system

$$N_n\varphi_n(t_{nj}) \equiv (a^2(t_{nj})-b^2(t_{nj}))\varphi_n(t_{nj}) + \frac{1}{\pi^2}Q^n_n\varphi_n(t_{nj}) = g_n(t_{nj}) \qquad (4.3.14)$$

$$j = 1,2,\ldots,n.$$

Recall that the nodes are taken on the arc coordinate. To illustrate the nature of this system consider the quadrature approximation

$$f_n(s) = \int_L \frac{P_n f(t)}{t-s}dt = \int_L \frac{\sum_{k=0}^{n-1} f(t_k)\varphi_k(t)}{t-s}dt$$

where $\varphi_k(t)$ is the piecewise linear function defined in Section 3.5. Suppose $s=t_j$. The j^{th} term in the above sum vanishes since it is

$$\frac{f(t_j)}{h}\left\{\int_{t_{j-1}}^{t_j} \frac{t-t_{j-1}}{t-t_j}dt + \int_{t_j}^{t_{j+1}} \frac{t_{j+1}-t}{t-t_j}dt\right\}$$

$$= \frac{f(t_j)}{h}\left\{-\int_0^h \frac{h-u}{u}du + \int_0^h \frac{h-u}{u}du\right\} = 0,$$ by the substitution $u = t_j-t$ in the

first integral and the substitution $u = t-t_j$ in the second integral. It is convenient to consider the remaining terms in pairs, symmetric about the j^{th} term. For example the first pair is

$$\frac{f(t_{j-1})}{h}\left\{\int_{t_{j-2}}^{t_{j-1}} \frac{t-t_{j-2}}{t-t_j}dt + \int_{t_{j-1}}^{t_j} \frac{t_j-t}{t-t_j}dt\right\} + \frac{f(t_{j+1})}{h}\left\{\int_{t_j}^{t_{j+1}} \frac{t-t_j}{t-t_j}dt + \int_{t_{j+1}}^{t_{j+2}} \frac{t_{j+2}-t}{t-t_j}dt\right\}$$

Similar substitutions to those above reduce this term to

$$\{f(t_{j+1})-f(t_{j-1})\}\{(2\log 2-1)+1\}$$

Thus we obtain a scheme of "symmetric pairing" similar to that obtained by Bareiss and Neuman, mentioned previously. Note that in the present case the interval of integration is the closed contour L and thus, whether in this form or in the equivalent parametric form considered by Gabdulhaev, the functions may be considered as periodic over the range of integration. Thus we may make the convention

$$f(t_{-j}) = f(t_{n-j}) = f(t_{2n-j}).$$

Thus we may write

$$f_n(t_j) = \int_L \frac{P_n f(t)}{t-t_j}dt = \left\{f(t_{j+1})-f(t_{j-1})\right\} 2 \log 2$$

$$+ \sum_{k=2}^{\max(j,n-j)} \{f(t_{j+k})-f(t_{j-k})\}\{(k+1)\log\frac{k+1}{k}-(k-1)\log\frac{k}{k-1}\}$$

Substituting from this expression into the system (4.3.14), where the term involving Q_n^n is evaluated from (4.3.3) and (4.3.4), we obtain a linear system to be solved for $\varphi_n(t_{nj})$, $j = 1,\ldots,n$, which involves only function evaluations of the given functions.

It is noted that this system is considerably more complicated than that proposed by Gabdulhaev, but the previous analysis shows that even in this more complex system only slow convergence is guaranteed. However it is assured that the solution does converge to the correct result and this does not follow for Gabdulhaev's system.

4.4 Concluding Remarks

The original aim of this investigation was simply to attempt to extend Anselone's approximation theory to the case of linear singular integral equations. However it was clear that the occurrence of the principal value integrals posed special problems and the collectively compact operator approximation theory was investigated in order to determine whether this might be extended to singular integrands.

The Hilbert transform exists in the space of Hölder continuous functions but approximations of such operators obtained by piecewise interpolation of the operand do not necessarily converge, even pointwise, in the Hölder space. This result poses a serious problem for a direct approximation

method and an analysis of Gabdulhaev's direct method shows that his investigation of the convergence of the solutions obtained by polygonal approximations cannot be considered as satisfactory. This argument is supported by an investigation of the general theory of approximation methods of Kantorovich and Akilov which again shows that the required conditions are not satisfied.

Hence it is concluded that it is necessary to follow the method of theoretical investigation and analytical solution of this singular integral equation and reduce it to the more tractable case of a weakly singular Fredholm equation. The theory of this reduction is well known from the standard texts, although it was found necessary to slightly strengthen the usual hypothesis on the density function in order to obtain a consistent equation. Finally it is shown that the approximate solutions obtained from an approximation method based on polygonal approximations of the numerator of the reduced equation do converge in principle but more precise computational details are left to a subsequent investigation of some particular example.

Apart from computational experiments with the linear system suggested in the previous section, an important extension to this investigation would be to the case where the interval of integration is a non-closed are or finite interval on the real line. In this case the principal value integral preserves the Holder condition in the interior of the interval by the Privalov theorem but the transformed function may become unbounded at the end-points. Equations of this type are solved analytically by Muskhelishvili and Pogorzelski and the nature of the unboundedness of the solution at the end-points can be bounded. Thus it may be possible to

introduce a special norm or transformation to take account of this effect and possibly obtain a "uniform" approximation within these limitations.

Other generalizations posing additional problems include systems of linear singular integral equations, multi-dimensional singular integral equations and non-linear equations.

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. Abstracts

This paper investigates the application of the collectively compact operator approximation theory to singular integral equations in a Hölder space. It is shown that the convergence of the approximate solutions of the singular integral equation, obtained by a direct method, cannot be guaranteed by this theory. However, a direct approximation of the reduced equation is shown to be convergent in the desired space.

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